Self-affine tiles and connectedness

by

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Radix expansion on $\mathbb{R}$: Let $q \geq 2$ be an integer

$$x = u + v = \Big(\sum_{k=-N}^{0} + \sum_{k=1}^{\infty}\Big) q^{-k} a_k, \qquad a_k \in \{0, 1, \cdots q-1\}.$$

The set of all the v's is $T = [0, 1]$, and the set of all the v's is $\mathbb{Z}$. T tiles $\mathbb{R}$ with the tiling set $\mathbb{Z}$.

Self-affine tile– Let $(A, \mathcal{D})$ be an (integral) affine pair, i.e.,

$A \in M_d(\mathbb{Z})$ and is expanding, $|\det A| = q$;

$\mathcal{D} = \{0 = d_1, \cdots, d_N\} \subset \mathbb{Z}^d$.

Consider

$$T = \{\sum_{j=1}^{\infty} A^{-j} d_j : \; d_j \in \mathcal{D}\},$$

T is called a *self-affine set*. It satisfies

$$AT = T + \mathcal{D}$$

Basic Theorem (*Bandt*) If $\#\mathcal{D} = |\det A|$ and $T^o \neq \emptyset$, then there exists a discrete set $\mathcal{T}$ such that

(i) $\bigcup_{t \in \mathcal{T}} (T + t) = \mathbb{R}^d$;

(ii) $(T + t)^o \bigcap (T + t')^o = \emptyset$ for all $t, t' \in \mathcal{T}, \; t \neq t'$.

Definition. We therefore call a self-affine set a *self-affine tile* if $T^o \neq \emptyset$ and $\mathcal{T}$ a (translational) tiling set.

- late 80's Thurston, Kenyon: *certain tilings imply self-affine tiles*;

- in the 90's – Bandt: *introduced the fractal concept*;

  Gröchenig & Madych : *connection with multi-resolution in wavelet theory*;

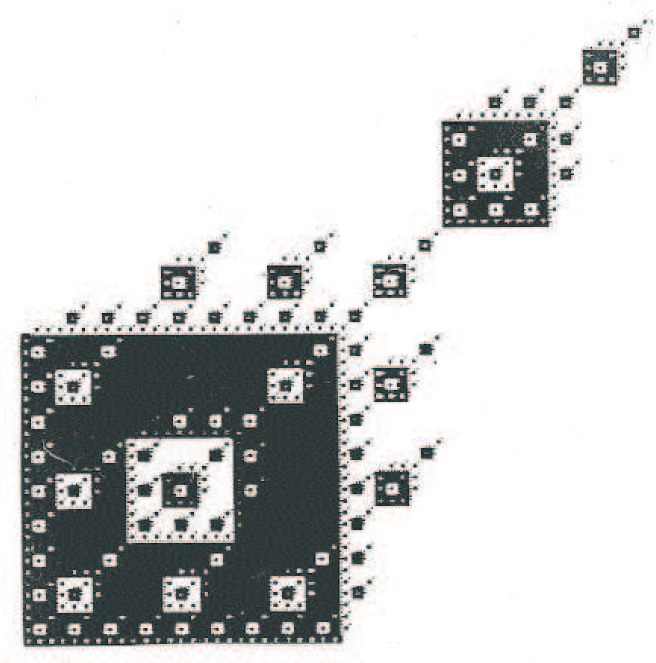
  Lagarias & Wang: *extended Kenyon's idea and set up the basic theory*;

- Problems concerning self-affine tiles:
  - characterization the tile digit set $\mathcal{D}$;
  - properties of the tiling set;
  - geometry of the boundary and dimension;
  - topological properties: *connectedness, disk-likeness*;
  - spectral sets and tiles: *Fugledge problem*

The connectedness of the tiles:

$$A = 3I = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},$$

$$\mathcal{D} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \end{bmatrix} \right\}$$



Bandt & Gelbbrich (1994), Hacon et al(1994), Bandt & Wang, Akiyama & Thuswaldner, Ngai, Tang, Luo, Rao, Leung, Kirat & L .

Proposition On $\mathbb{R}$, let $A = [q]$, $|q| \geq 2$ and $\mathcal{D} \subset \mathbb{Z}$ has $|q|$-digits. Then $T(A, \mathcal{D})$ is connected if and only if , up to a translation, $\mathcal{D} = \{0, a, \cdots, (|q| - 1)a\}$ for some $a > 0$.

Q: What is the connectedness of T in $\mathbb{R}^d$ if we take $\mathcal{D} = \{0,\ v, \cdots,\ (q-1)v\}$, $q = |\det A|$, $v \in \mathbb{Z}^d$ (call it *consecutive collinear (CC) digit sets*)?

In order to have a tile, it is necessary that $\{v, Av, \cdots, A^{d-1}v\}$ to be independent.

Connectedness for CC digit sets (*Kirat & L*)

For a self-affine pair $(A, \mathcal{D})$, let

$$\mathcal{E} = \{(d_i, d_j) : \ (T + d_i) \cap (T + d_j) \neq \emptyset, \ d_i, d_j \in \mathcal{D}\}.$$

A basic criterion. T is connected if and only if for any $d_i, d_j \in \mathcal{D}$, there exists $d_i = d_{i_1}, d_{i_2}, \cdots, d_{j_k} = d_j$ such that $(d_{j_i}, d_{j_{i+1}}) \in \mathcal{E}$.

Note that $(d_i, d_j) \in \mathcal{E}$ if and only if

$$d_i - d_j = \sum_{k=1}^{\infty} A^{-k} v_k, \quad \text{for some } \ v_k \in \mathcal{D} - \mathcal{D}.$$

It follows that

A condition for CC digit set. T is connected if and only if

$$\sum_{n=0}^{\infty} b_n A^{-n} v = 0,$$

with $b_0 = 1, \ b_n \in \{0, \pm 1 \cdots, \pm(q-1)\}, \ \ n \geq 1.$

Using the above, we can formulate an **algebraic criterion** for the connectedness.

Let $A \in M_d(\mathbb{Z})$ be expanding and the c.p. is

$$f(x) = x^d + c_{d-1}x^{d-1} + \cdots + a_1 x + q.$$

Definition. $f(x)$ is said to have the *height reducing property* (HRP) if there exists an $h(x) \in \mathbb{Z}[x]$ such that

$$g(x) = h(x)f(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_1 x \pm q$$

with $|b_i| \leq q - 1$.

Theorem 1. Let $(A, \mathcal{D})$ be as the above. Then T is connected if $f(x)$ has the HRP.

Main idea: By the Hamilton-Cayley theorem, $g(A) = 0$, hence

$$-A = b_{k-1}A^{-1} + \cdots + (q-1)A^{-k} + A^{-k} := p(A) + A^{-k}$$

Therefore

$$0 = A + p(A) + A^{-k} = A + \sum_{\ell=1}^{\infty}(-1)^{\ell+1}p(A^{-\ell k}),$$

and the above connectedness criterion applies.

Theorem 2. The HRP holds for $d \leq 4$. Hence on $\mathbb{R}^d$, $d \leq 4$ the tiles with CC digit sets are connected.

($n \leq 3$: *Kirat, Rao & L;* $d = 4$*: Akiyama & Gjini*)

Conjecture: All expanding polynomials has the HRP.

Some evidence :

- *Garsia* : $g(x) = h(x)f(x) = b_k x^k + \cdots + b_1 x \pm q, \quad |b_i| \leq q.$

- *Kirat, Rao & L* : improve to $|b_i| \leq q - 1$.

- We introduce a new setup (constructive) to study the HRP (*He, Kirat & L*)

Given $f(x) = x^d + c_{d-1}x^{d-1} \cdots + c_1 x + q$ (not necessary expanding).

Consider $(B, \mathcal{C})$: $\mathcal{C} = \{0, e_1, \cdots, (|q|-1)e_1\}$ and B is the companion matrix

$$B = \begin{bmatrix} 0 & \cdots & 0 & -q \\ 1 & & & -c_1 \\ & \ddots & & \vdots \\ 0 & & 1 & -c_{d-1} \end{bmatrix}.$$

Let $\mathcal{J}_0 = \{\pm e_1, \cdots, \pm e_d\}$;

$\mathcal{J}_n = \{u : \ Bu \in \mathcal{J}_{n-1} + (\mathcal{C} - \mathcal{C})\}$ and

$\mathcal{J} = \bigcup_n \mathcal{J}_n$.

Theorem 3. For $1 \leq s \leq |q| - 1$. Then $se_1 \in \mathcal{J}$ if and only if there exists $h(x)$ such that

$$g(x) = h(x)f(x) = sx^k + b_{k-1}x^{k-1} + \cdots + b_1 x \pm q$$

where $b_i \leq |q| - 1$. (*We call this s-HRP.*)

Main idea : $se_1 \in \mathcal{J}$ implies $sB^n e_1 = \pm e_j - \sum_{i=0}^{n-1} b_i B^i e_1$ with $|b_i| \leq |q| - 1$. Let

$$g(x) = sx^n + b_{n-1}x^{n-1} + \cdots + (b_{j-1} \pm 1)x^{j-1} + \cdots + b_0.$$

Then $g(B) = 0$, so that $g(x) = h(x)f(x)$. That $g(0) = h(0)q$ implies that the e_j must be e_1 and $(b_0 \pm 1) = q$. Hence $f(x)$ has the s-HRP.

Theorem 4. If $f(x)$ is expanding, then $\mathcal{J}$ is a finite set and $se_1 \in \mathcal{J}$ for some $1 \leq s \leq |q| - 1$.

The algorithm: Make use of: $u \in \mathcal{J}_n$ if and only if

$$u = B^{-1}(w + je_1) \in \mathbb{Z}^d$$

for some $w \in \mathcal{J}_{n-1},\ 0 \leq |j| \leq |q| - 1$.

Then search for u such that $u = (s, 0, \cdots, 0)$.

• Computer experiments showed that the "non-expanding" polynomial with the HRP is rare, and they are most often 1-HRP.

By using the algorithm, we can show

Proposition 5. $f(x) = x^2 + px + q$ has HRP if and only if $|p| \leq q$ when $q > 0$, and $|p| \leq |q| - 1$ when $q < 0$.

Note that such $f(x)$ is expanding if and only if $|p| \leq q$ when $q > 0$, and $|p| \leq |q| - 2$ when $q < 0$.

Planar Self-affine tiles :

Theorem 6 (*Leung & L, 07*) Let $f(x) = x^2 + px + q$ be the c.p. of an expanding $A \in M_2(\mathbb{R})$ and let $\mathcal{D}$ be a CC digit set. Then T is *disk-like* if and only if $2|p| \leq |q+2|$.

This extends a result of *Akiyama & Thuswaldner* on canonical number systems : the algebraic number α with minimal polynomial $f(x) = x^2 + px + q$ with $-1 \leq p \leq q, \ q \geq 2$.

The proof makes use (*Bandt, Gelbrich & Wang*): *For T a connected $\mathbb{Z}^2$-tile, if T has no more than six neighbors, then T is disk-like.*

The checking of the neighborhoods is to use the radix expansion and the Hamilton-Cayley theorem.

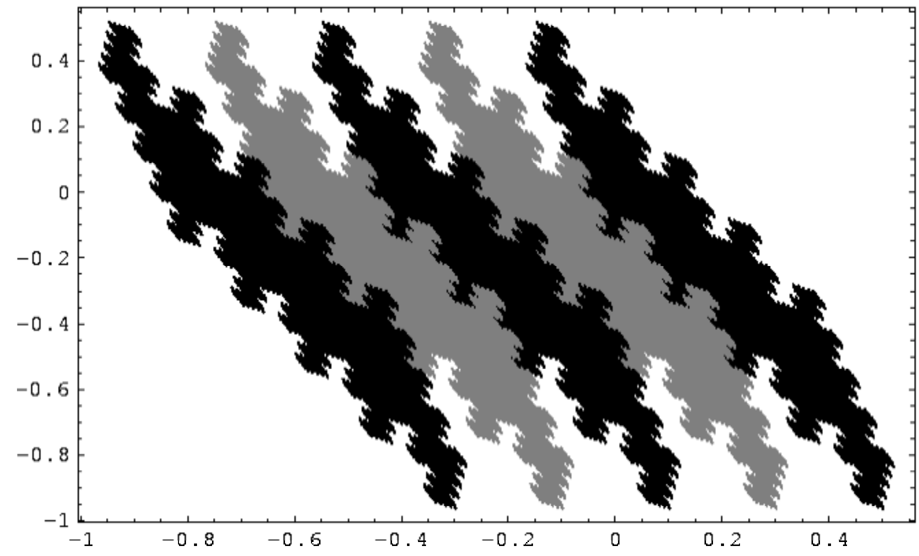


FIGURE 1. A disklike tile, $f(x) = x^2 + 3x + 5, \quad 2p < q + 2$

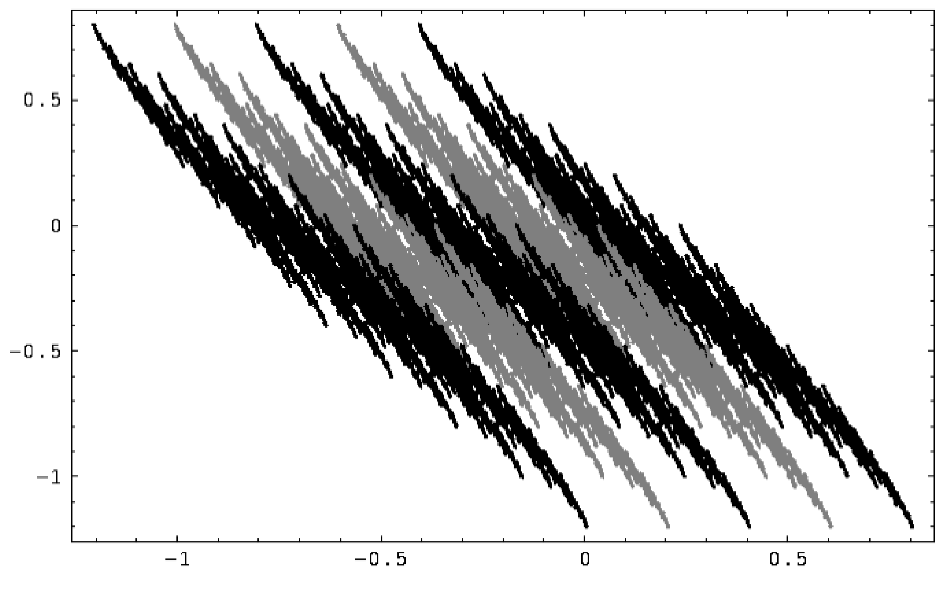


FIGURE 2. A non-disklike tile, $f(x) = x^2 + 4x + 5, \quad 2p > q + 2$

Digit sets that are non-collinear *(Deng & L).*

Theorem 7. *Assume* $p, q \geq 2$. *For*

$$A = \begin{bmatrix} p & 0 \\ -a & q \end{bmatrix}, \quad \mathcal{D} = \Big\{ \begin{bmatrix} i \\ j \end{bmatrix} : 0 \leq i \leq p-1, \ 0 \leq j \leq q-1 \Big\}.$$

Then T *is a connected if and only if* $\mid \dfrac{a}{q(q-1)} \mid \leq 1$,
and T *is disk-like if and only if* $\mid \dfrac{a}{q(q-1)} \mid < 1$.

Here $\quad T = \Big\{ \begin{bmatrix} p(\mathbf{i}) \\ ar(\mathbf{i}) + q(\mathbf{j}) \end{bmatrix} : \quad 0 \leq i_n < p, \ 0 \leq j_n < q \Big\}.$

with

$$p(\mathbf{i}) = \sum_n \frac{i_n}{p^n}, \quad r(\mathbf{i}) = \sum_n r_n i_n, \quad q(\mathbf{j}) = \sum_n \frac{j_n}{q^n}.$$

and $r_n = \begin{cases} (p^{-n} - q^{-n})/(q-p), & \text{if } p \neq q, \\ n/p^{n+1}, & \text{if } p = q. \end{cases}$

T is in between the two vertical line $x = 0$ and $x = 1$, and the two sides are line segments.

Example:

$$A = \begin{bmatrix} 2 & 0 \\ -a & 2 \end{bmatrix}, \quad \mathcal{D} = \left\{ \begin{bmatrix} i \\ j \end{bmatrix} : i,\ j = 0, 1 \right\}$$

The connectedness condition is $|a/2| \leq 1$.

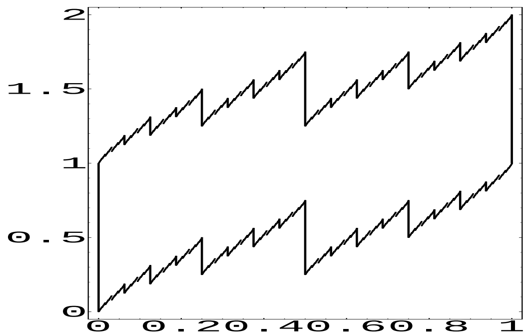


FIGURE 3. $a = 1$

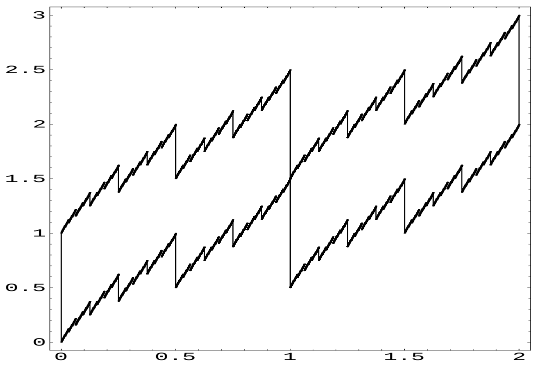


FIGURE 4. $a = 2$

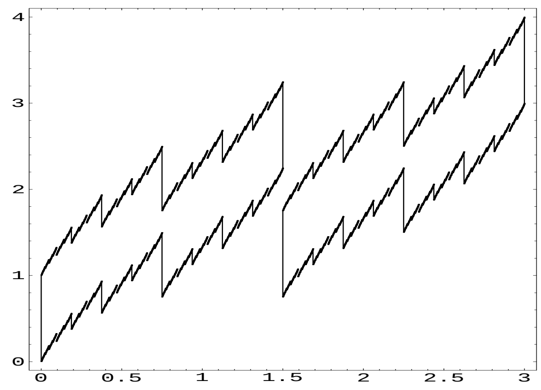


FIGURE 5. $a = 3$

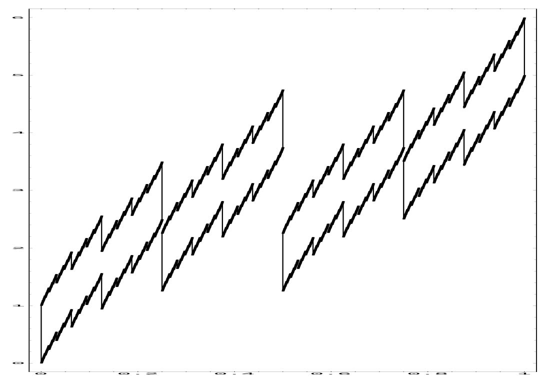


FIGURE 6. $a = 5$

By increasing a, there are more components.

Theorem 8. Let T be the self-similar tile as in Theorem 7. Suppose $q^{m-1} < \frac{a}{q(q-1)} \leq q^m$. Then T has p^m connected components.

Q. What is the tiling and connectedness property with the the above A and other simple arrangements of the digit sets?