Boundary parametrization of self-similar tiles.

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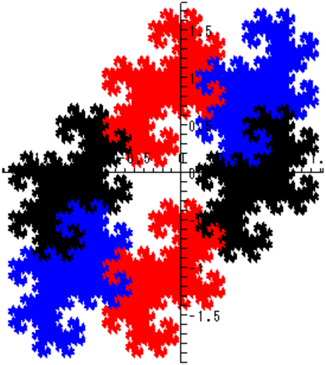
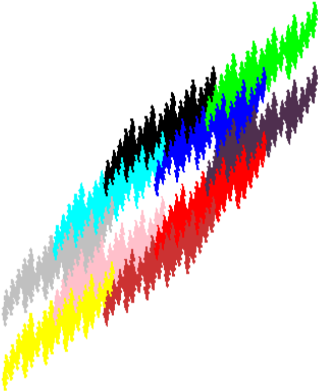
Introduction



$\alpha = -1 + i$



$\alpha = -2 + i$

$$T = \left\{ \sum_{j \geq 1} a_j \alpha^{-j} ; a_j = 0, \dots, |\alpha|^2 - 1 \right\}.$$

Boundary of attractors

Let $T \subset \mathbb{R}^2$ attractor of an IFS with open set condition. Then

- T connected $\Rightarrow \partial T$ is connected (Akiyama-Luo-Thuswaldner).
- T connected $\Rightarrow \partial T$ is a locally connected continuum (Tang).

Hence, if T is connected, there exists $f : [0,1] \to \partial T$ continuous surjection.

Self-affine digit tiles

T is *integral self-affine digit tile* if

$$\mathbf{A}T = \bigcup_{a \in \mathcal{D}} (T + a),$$

where

- $\mathbf{A} \in \mathbb{Z}^{d \times d}$ *expanding* matrix : $|\mathrm{sp}(\mathbf{A})| \subset (1, +\infty)$.
- $\mathcal{D} \subset \mathbb{Z}^d$ is a *complete residue system* of $\mathbb{Z}^d / \mathbf{A}\mathbb{Z}^d$.

Under these conditions, $T = \overline{T^o}$.

$$T = \left\{ \sum_{j \geq 1} \mathbf{A}^{-j} a_j ; a_j \in \mathcal{D} \right\}.$$

Existence of tiling set

Theorem (Lagarias-Wang, Gröchenig-Haas)

T integral self affine digit tile. Then there is $\mathcal{J} \subset \mathbb{R}^d$ such that

$$T + \mathcal{J} \quad \textit{is a tiling of } \mathbb{R}^d.$$

No information on the structure of $\mathcal{J}$.

Boundary of $\mathbb{Z}^d$-self-affine tiles

Suppose $\mathcal{J} = \mathbb{Z}^d$. Neighbors of the tile T :

$$\mathcal{S} = \{s \in \mathbb{Z}^n \setminus \{0\}; T \cap (T+s) \neq \emptyset\}$$

is finite.

- By the tiling property,

$$\partial T = \bigcup_{s \in \mathcal{S}} \underbrace{T \cap (T+s)}_{B_s}.$$

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- Since $\mathbf{A}T = \bigcup_{d \in \mathcal{D}} (T+d)$, we have

$$\begin{aligned} B_s &= \bigcup_d \mathbf{A}^{-1}(T+d) \\ &\quad \cap \bigcup_{d'} \mathbf{A}^{-1}(T+d'+\mathbf{A}s) \\ &= \bigcup_{d,d'} \mathbf{A}^{-1}(\underbrace{T \cap (T+d'+\mathbf{A}s-d)}_{=B_{s'}}+d). \end{aligned}$$

Boundary automaton

- Let $G(\mathcal{S})$ be the following directed automaton :
  - states : $s \in \mathcal{S}$;
  - edges : for $s, s' \in \mathcal{S}$ and $d, d' \in \mathcal{D}$,

$$s \xrightarrow{d|d'} s' \text{ iff } \mathbf{A}s + d' - d \;=\; s'.$$

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- Then

$$B_s = \bigcup_{s \xrightarrow{d|d'} s'} \mathbf{A}^{-1}(B_{s'} + d),$$

$$\partial T = \bigcup_{s \in \mathcal{S}} B_s$$

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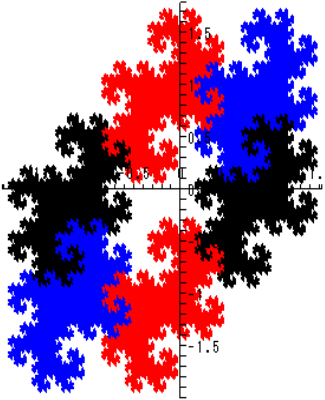
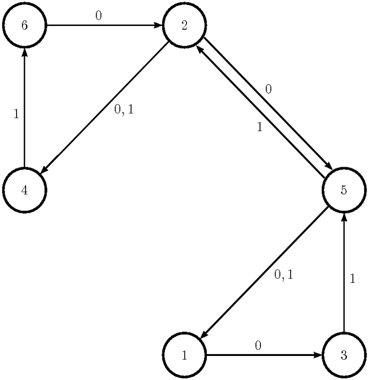
- $G(\mathcal{S})$ is a finite automaton and can be computed algorithmically.

Example : boundary automaton of Knuth tile

$\mathbf{A}, \mathcal{D}$ corresponding to $\alpha = -1 + i$, $D = \{0, 1\}$.





Boundary language

- Boundary points : $x = \sum_{j=1}^{\infty} \mathbf{A}^{-j} d_j \in T \cap (T+s)$ iff there is an infinite walk $w = (d_1, d_2, \ldots)$ in $G(\mathcal{S})$

$$s \xrightarrow{d_1} s_1 \xrightarrow{d_2} s_2 \xrightarrow{d_3} \ldots$$

- Thus the language of the automaton is the language of the boundary $\mathcal{L}(\partial T) \subset \mathcal{D}^{\mathbb{N}}$.

$$\begin{array}{rcl} \psi : \mathcal{L}(\partial T) \to & \partial T \\ w = (d_1, d_2, \ldots) & \mapsto & \mathbf{A}^{-1} d_1 + \mathbf{A}^{-1} d_2 + \ldots \end{array}$$

  is surjective.

- Aim : find ϕ such that $[0,1] \xrightarrow{\phi} G(\mathcal{S}) \xrightarrow{\psi} \partial T$ is continuous surjection.

Boundary substitution

- Let $\mathbf{L}$ be the incidence matrix of the automaton : $\mathbf{L} = (l_{i,j})$ with
$$l_{i,j} = \#\{\text{edges } i \to j\}.$$
If $\mathbf{L}$ is *primitive*, the automaton $G(\mathcal{S})$ describes a *Dumont-Thomas* substitution. We can construct a parametrization of ∂T using a number system with basis β, the Perron Frobenius eigenvalue of $\mathbf{L}$.

- In general, a primitive boundary subautomaton has to be found.

Canonical number system

$P = x^2 + Ax + B \in \mathbb{Z}[x]$ irreducible, α root of P.

- α is a *canonical number system* if every $x \in \mathbb{Z}[\alpha]$ has a unique α-representation

$$x = d_0 + d_1\alpha + \ldots + d_l\alpha^l,$$

  with integer digits $d_0, \ldots, d_l \in D := \{0, \ldots, |B| - 1\}$.
- This is the case iff $-1 \leq A \leq B \geq 2$.

Associated tiling

- Embedding into the plane :

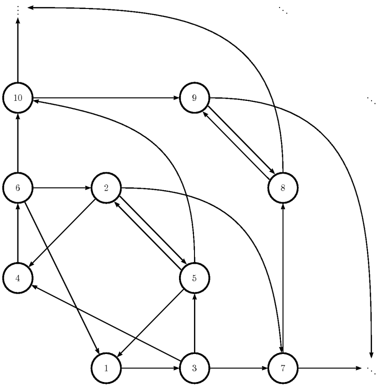
$$\begin{array}{ccc} \mathbb{Q}(\alpha) & \to & \mathbb{R}^2 \\ a+b\alpha & \mapsto & \begin{pmatrix} a \\ b \end{pmatrix}. \end{array}$$

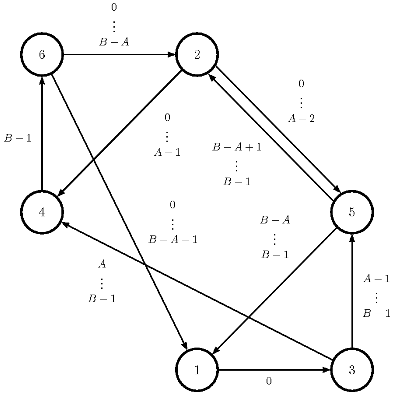
- The multiplication by α is given by the *expanding* matrix :

$$\mathbf{A} = \begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix}.$$

- $\left\{ \sum_{j\geq 1} \mathbf{A}^{-j} \begin{pmatrix} d_j \\ 0 \end{pmatrix}; d_j = 0, \dots, B-1 \right\} + \mathbb{Z}^2$ is a tiling.

Boundary automaton and primitive subautomaton G





Main result

Theorem

Let T be a quadratic CNS tile. Let β be the Perron Frobenius eigenvalue of the incidence matrix of the subautomaton. Then there exists $f : [0,1] \to \partial T$ continuous onto mapping and an hexagon $Q \subset \mathbb{R}^2$ with the following properties. Let $T_0 := Q$ and

$$\mathbf{A}T_n = \bigcup_{a \in \mathcal{D}} (T_{n-1} + a).$$

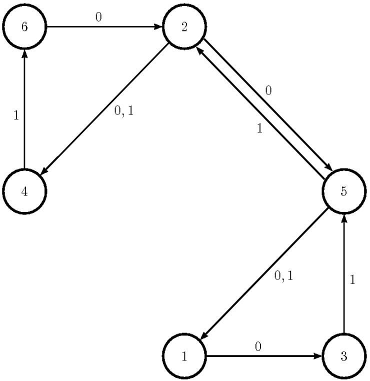
be the sequence of approximations of T associated to Q. Then :

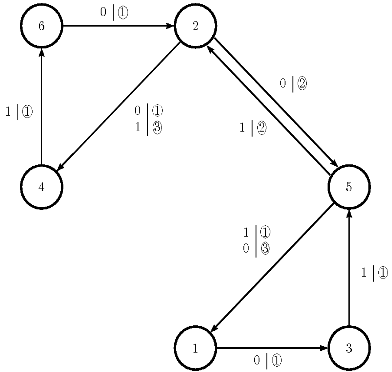
(1) $\lim_{n\to\infty} \partial T_n$ *(Hausdorff metric).*

(2) *For all $n \in \mathbb{N}$, ∂T_n is a polygonal simple closed curve.*

(3) *Denote by V_n the set of vertices of ∂T_n. For all $n \in \mathbb{N}$, $V_n \subset V_{n+1} \subset f(\mathbb{Q}(\beta) \cap [0,1])$ (i.e., the vertices have $\mathbb{Q}(\beta)$-adresses in the parametrization).*

Choice of substitution : ordered automaton G^o





$$\sigma: \quad 1 \to 3 \quad 2 \to 454 \quad 3 \to 5$$
$$4 \to 6 \quad 5 \to 121 \quad 6 \to 2$$

Parametrization of the automaton

- Let $\mathbf{L}$ the incidence matrix and u be defined by

$$\mathbf{L}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} = \beta \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}$$

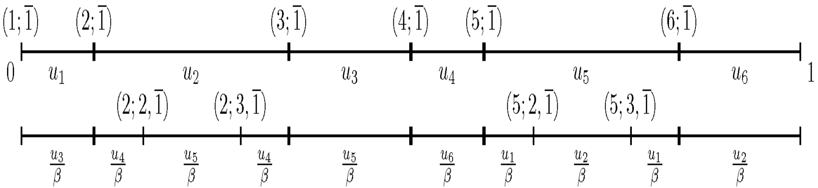
  and $u_1 + \ldots + u_6 = 1$. We have $u_j > 0$ and $u_j \in \mathbb{Q}(\beta)$.
- We now define $\Phi : G^o \rightarrow [0, 1]$.

Mapping $\Phi : G^o \to [0,1]$

A walk in G^o has the form

$$(i; o_1, o_2, o_3, \ldots)$$

where i is the initial state and (o_i) a sequence of orders.



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Properties of $\Phi : G^o \longrightarrow [0,1]$

- The above procedure defines :

$$\Phi : \begin{array}{ccc} G^o & \longrightarrow & [0,1] \\ v = (i; w = (o_1, o_2, \ldots)) & \mapsto & x(v) \end{array}$$

- The identifications are trivial : $\Phi(v) = \Phi(v')$ iff
  - $v = (i; o_{max}, o_{max}, \ldots), v' = (i+1; 1, 1, \ldots)$, or
  - $v = (v_0, o_{max}, o_{max}, \ldots)$ and $v' = (v_0, 1, 1, \ldots)$.
- We write $v \sim v'$ if $\Phi(v) = \Phi(v')$. Then

$$\tilde{\Phi} : G^o / \sim \longrightarrow [0,1]$$

becomes a bijection.

Boundary parametrization

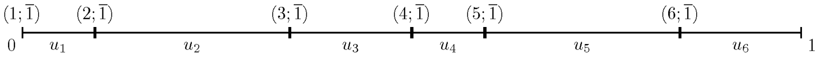
- We have a natural transition mapping : $G^o \xrightarrow{T} G(\mathcal{S}) \xrightarrow{\psi} \partial T$.
- **Compatibility Lemma**:

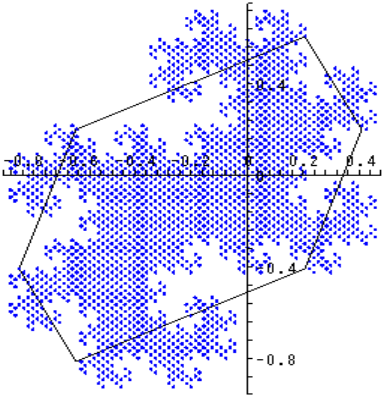
$$v \sim v' \Rightarrow \psi(T(v)) = \psi(T(v')).$$

  This is by choice of substitution. It can be checked by automata.
- **Lemma**. $f : [0,1] \to G^o/\sim \to \partial T$ *is continuous, surjective. Moreover,* $f(0) = f(1)$.

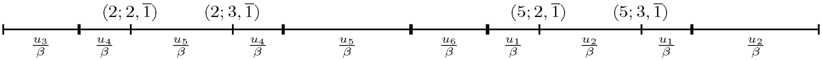
Boundary 1st approximation.

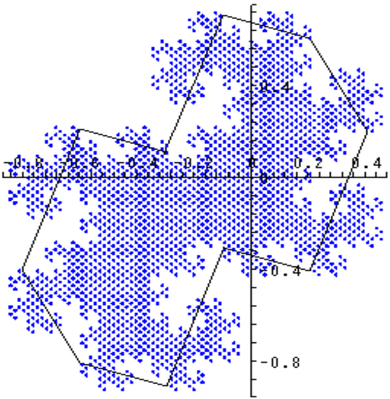
Lemma *The points* $f(0), f(u_1), f(u_1 + u_2), \ldots$ *form a hexagon* Q.

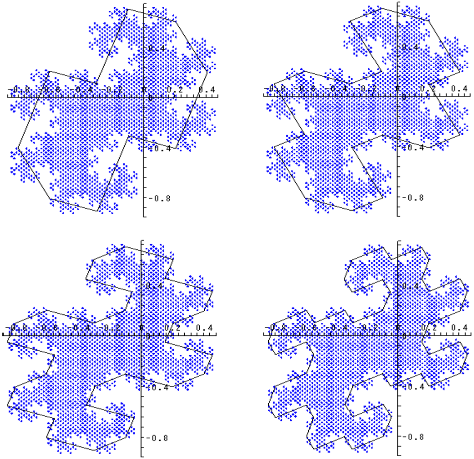




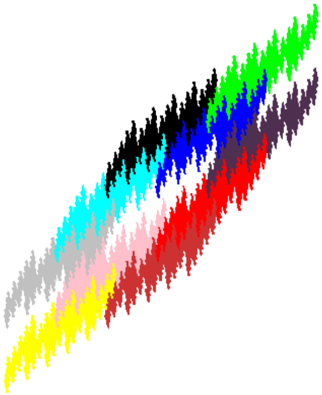
Boundary 2nd approximation.

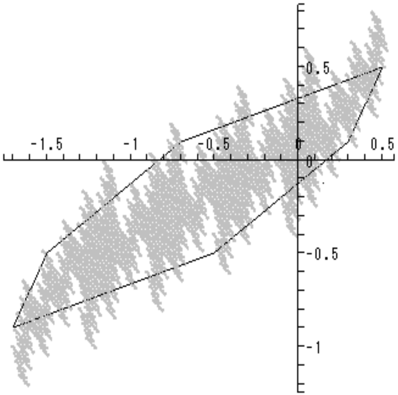




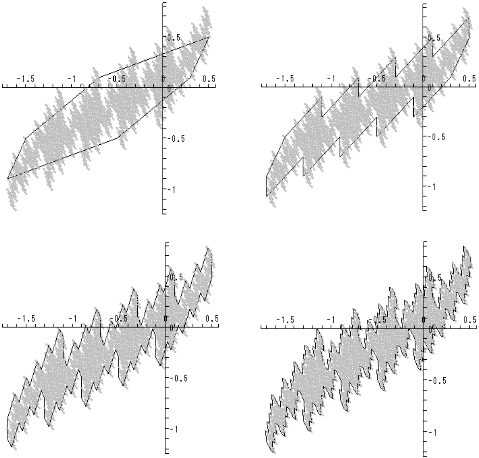


0.4
-0.4
-0.8

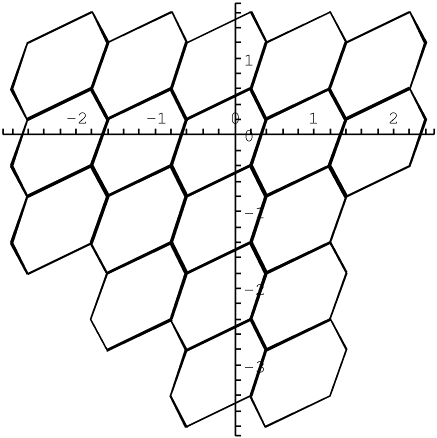



-1.5
-1
-0.5
0
0.5
0.5
-0.5
-1

$-2+i$.

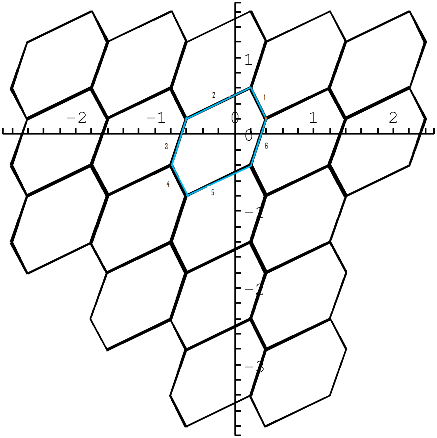


Lemma. $Q + \mathbb{Z}^2$ is a tiling of $\mathbb{R}^2$.

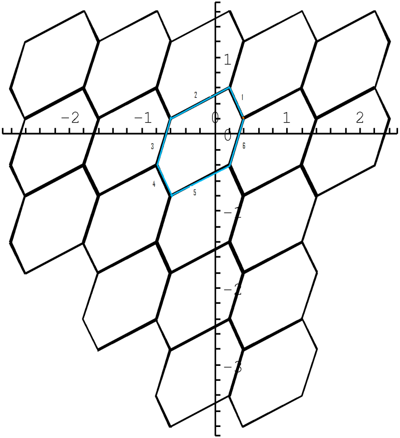


Relation to the substitution σ

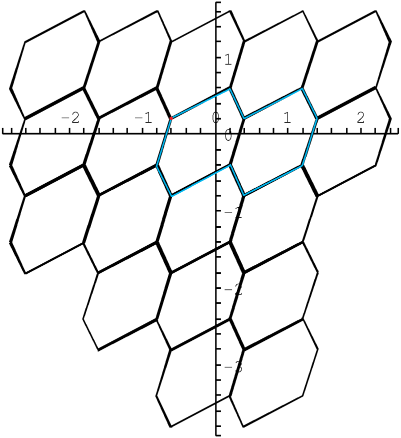
Consider oriented paths $p : \{1,2,3,4,5,6\}^{\mathbb{N}} \to \mathbb{R}^2$. We have $\partial Q = p(123456)$.



Iterations of σ

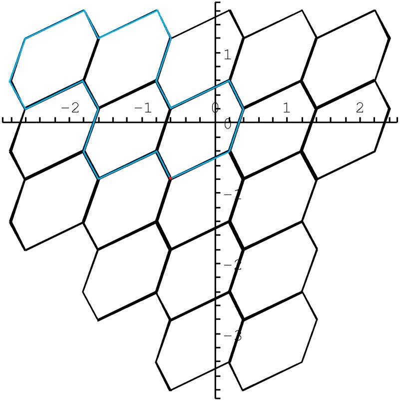


$p(123456)$.

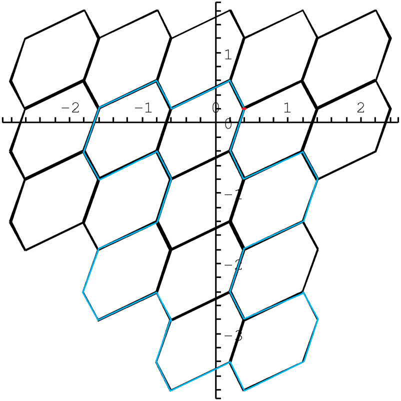


$p(\sigma(123456)) = p(3454561212)$.

Iterations of σ



$p(\sigma^2(123456))$



$p(\sigma^3(123456))$.

Lemma. *Let* $T_0 := Q$ *and* (T_n) *the associated sequence of approximations. Then* $\mathbf{A}^{-n}p(\sigma^n(123456)) = \partial T_n + k_n$.

Further results and work

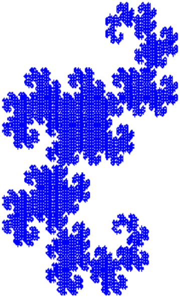
- For $2A - B < 3$, we have equivalence : $v \sim v'$ iff $\psi(v) = \psi(v')$. This can be checked by automaton. Hence

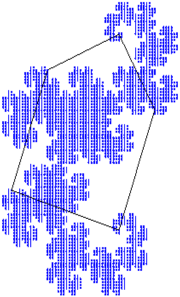
$$f : [0,1] \to \partial T$$

  is a continuous bijection, and ∂T is a simple closed curve (cf Akiyama-Thuswaldner).

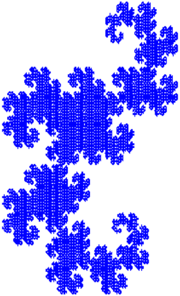
- For $2A - B \geq 3$, this does not hold. Identifications occur on ∂T that are not seen by the number system in β. The corresponding tiles are not homeomorphic to a disk.
- Extend to other classes of self-affine tiles, tiling by lattice or crystallographic group (example : Heighway dragon).

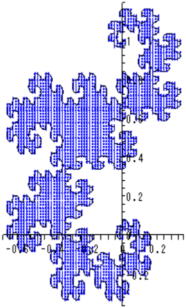
Heighway dragon





Heighway dragon





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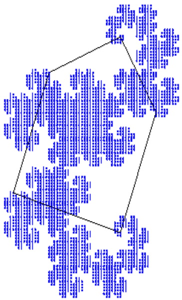

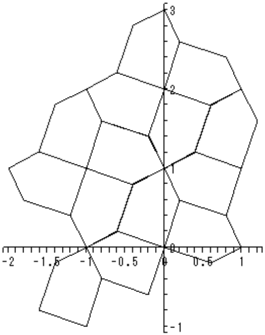

Non trivial identifications : cutpoints