It was difficult to decide whether to talk about Steinhaus' simultaneous tiling problem:

Is there some $S \subset R^2$ such that S meets every isometric copy of Z^2 in exactly one point?

In particular the open problem:

Can such a set S be Lebesgue measurable?

This problem seems to involve many not clearly understood issues in planar geometric measure theory and harmonic analysis.

and

Random fractals occurring in algorithmic randomness

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There has been a growing interaction between logicians, set theorists and computer scientists with analysts under the general umbrella of ”‘algorithmic randomness.” We will hear more about this from Jack Lutz who has done a lot of foundational work on this. It involves applications and modifications of techniques from all these areas for this new mixture and presents us with some opportunities for some new vibrant research directions.

Example: A Construction of Random Subsets of $C = \{0,1\}^N$

We illustrate our results by the following **basic example.** Consider the construction of a subset of $C = \{0,1\}^N$. First, build a random tree in $\{0,1\}^*$ as follows. Begin at the root. If a finite tree has been constructed, then for each end node of the tree, append only the left branch, append only the right branch, or append both with equal probability. Do this independently at each end node. Iterating, a random tree T_ω is constructed. Consider the random set $K_\omega \subset \{0,1\}^N$ consisting of all infinite paths through the tree T_ω and let P also be the corresponding measure induced on the coding space of this process.

P = 1/3 P = 1/3 P = 1/3

Several things are known about sets constructed in this manner. In particular, the following theorem was proven.

T.: (Barmpalias, Brodhead, Cenzer, Dashti, Weber)(2007) *For $P-a.e.$ K, $dim_{H,\rho}(K) = \log_2\left(\frac{4}{3}\right) =: \eta$, where $dim_{H,\rho}$ is Hausdorff dimension with respect to the ultrametric $\rho(\sigma,\tau) = \frac{1}{2^{|\sigma\wedge\tau|+1}}$.*

Actually, their theorem says for every P-random K, the dimension formula holds. They are consdering a somewhat different notion of randonmess, but for our pruposes we can consider our usual notion.

Much more is known. The following theorem gives us the exact Hausdorff measure function.

T.: (Mau-McL) *For P-a.e. K, $\mathcal{H}^{\eta}(K) = 0$. Moreover, for P-a.e. K, $0 < \mathcal{H}^{g}(K) < \infty$, where $g(t) = t^{\eta}|\log|\log t||^{2-\frac{\log 4}{\log 3}}$.*

Method of proof:

- Regard the procedure described above as producing a random subset of C = Cantor's middle $\frac{1}{3}$ set. This turns out to be a random recursive construction as described by Mauldin and Williams in TAMS, 295, 1986, 325-346. Applying results from this paper and results of Graf, Mauldin and Williams in Memoirs AMS no. 381, 1988, we find that $dim_{H,|\cdot|}(K) = \alpha = \frac{\log 4}{\log 3} - 1$ as a subset of C with the usual Euclidean metric $|\cdot|$. We also find $\mathcal{H}^\alpha(K) = 0$ and for $P - a.e.\ K$, $0 < \mathcal{H}^k(K) < \infty$, where $k(t) = t^\alpha |\log|\log t||^{1-\alpha}$. (Again, this is all with respect to the euclidean metric.)

- Transfer results to $\{0,1\}^N$. Use the coding map $\phi : \{0,1\}^{\mathbb{N}} \mapsto C$, $\phi(\sigma) = \sum_{i \geq 1} \frac{2\sigma(i)}{3^i}$. ϕ is bi-Hölder of order $d = \frac{\log 3}{\log 2}$:

$$\rho(\sigma,\tau)^d \leq |\phi(\sigma) - \phi(\tau)| \leq 3\rho(\sigma,\tau)^d.$$

- Apply geometric results about how Hausdorff dimension and measures transform: For $E \subset C$, $dim_{H,\rho}(E)) = d \cdot dim_{H,|\cdot|}(\phi(E))$ and

$$0 < \mathcal{H}^k(\phi(E)) < \infty,$$

if and only if

$$0 < \mathcal{H}^g(E) < \infty.$$

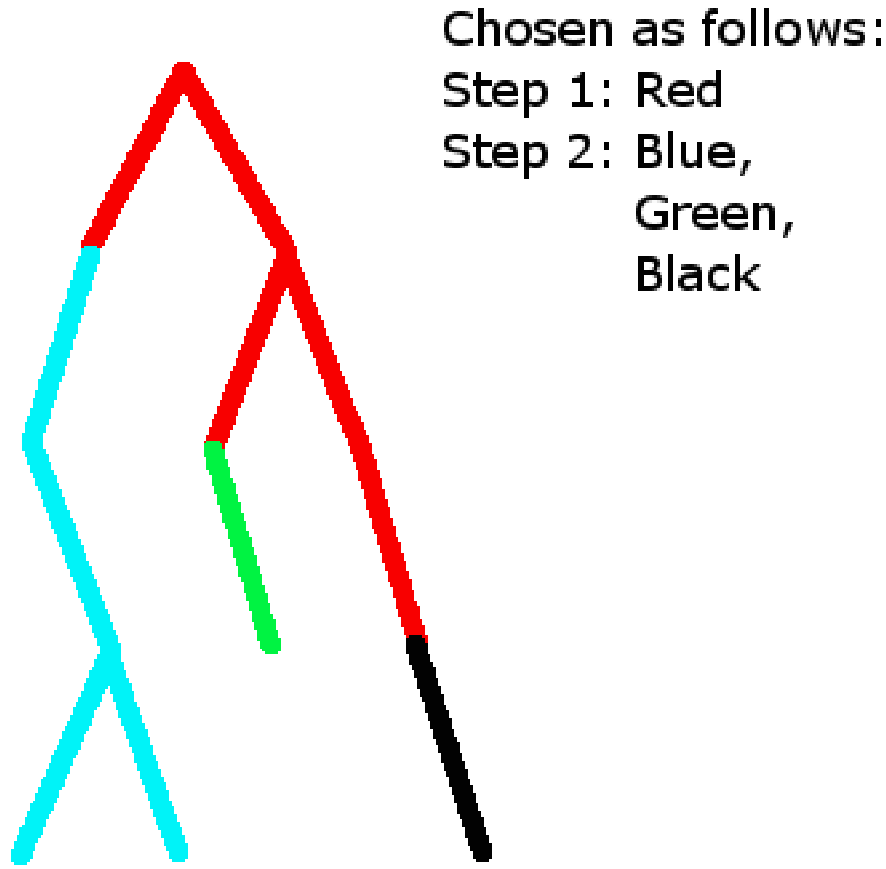
By the way, this transfer theorem certainly works for Hausdorff gauge functions of the form

$$g(t) = t^{\alpha} \cdot L(t),$$

where L is some product of powers of iterated logarithms. But I don't know if the theorem is true if $L(t)$ is simply a slowly varying function of Karamata.

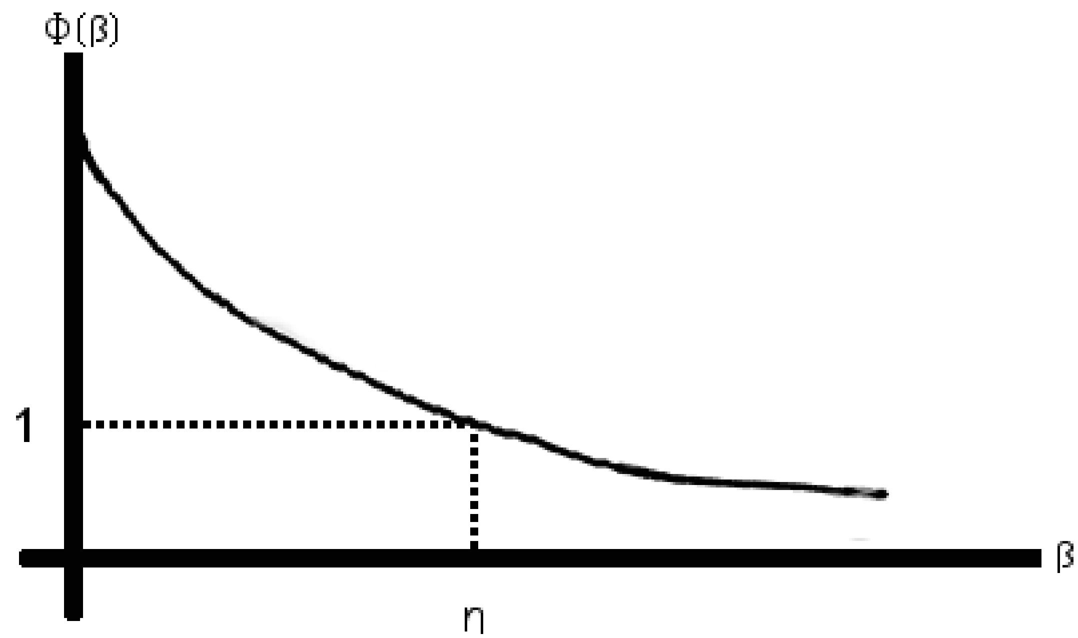
These theorems are special cases of general random recursions in R^N given by Graf-Mauldin-Williams. First let us state a more general version for 2 letter alphabets.

General results (stated for alphabets with 2 letters). Let $A = \{0, 1\}$ and let $F(A^*)$ denote the set of prefix free subsets of A^*, and let P be a probability distribution on $F(A^*)$. Then we generate a random subset of A^N using P. Thus, if the construction has generated a finite tree at some stage we independently append to each end node a random element of $F(A^*)$.


Chosen as follows:
Step 1: Red
Step 2: Blue,
Green,
Black

We find the Hausdorff dimension, η, with respect to the ultra metric as follows. Define $\Phi : [0,\infty) \to [0,\infty)$ by $\Phi(\beta) = E_{B\in F(A^*)}\left[\sum_{\sigma\in B}\left(\frac{1}{2^{|\sigma|}}\right)^{\beta}\right]$.

The geometric properties of Φ are indicated in the figure.



As will be indicated, the Hausdorff dimension of the random set is a. s. η, where $\Phi(\eta) = 1$. For the **basic example**, $\Phi(\beta) = \frac{1}{3}\frac{1}{2^\beta} + \frac{1}{3}\frac{1}{2^\beta} + \frac{1}{3}[\frac{1}{2^\beta} + \frac{1}{2^\beta}] = \frac{4}{3}\frac{1}{2^\beta}$. Solving $\Phi(\eta) = 1$, we find $dim_{H,\rho}(K) = \log_2(\frac{4}{3})$.

Here are some general theorems.

T.:

- If $\Phi(0) > 1$, $P(K \neq \emptyset) > 0$

- For $\eta := \inf\{\beta : \Phi(\beta) \leq 1\}$,
  $P(dim_{H,\rho}(K) = \eta | K \neq \emptyset) = 1$.

- If $P_{B \in F(A^*)}(\sum_{\sigma \in B} \left(\frac{1}{2^{|\sigma|}}\right)^{\eta} \neq 1) > 0$, then $P(\{\omega : \mathcal{H}^{\eta}(K(\omega)) = 0\}) = 1$.

Comment: The condition $\Phi(0) > 1$ (from branching processes) ensures us that with positive probability $K \neq \emptyset$ and is in fact a Cantor set.

T.: If P is supported on a finite set of trees and $P_{B \in F(A^*)}(\sum_{\sigma \in B} \left(\frac{1}{2^{|\sigma|}}^{\eta}\right) \neq 1) > 0$, then $P(0 < \mathcal{H}^{g}(K) < \infty | K \neq \emptyset) = 1$, where $g(t) = t^{\eta} |\log|\log t||^{1-\alpha}$, where $\alpha = \eta \log_3(2)$.

Comment: It is not necessary that P be supported on finitely many tress but the exact dimension function is not known then.

I want to indicate some hueristics for how one arrives at the exact Hausdorff dimension function g.

We are first interested in the functions of the form $g(t) = t^\alpha$ and then in those of the form $g(t) = t^\alpha L(t)$ where L is slowly varying:

$$\forall \lambda > 0, \lim_{t \to 0} \frac{L(\lambda t)}{L(t)} = 1.$$

For the finite alphabet spaces with the standard ultra metric, there is another characterization due to Jack Lutz of Hausdorff dimension using s-gales. We consider $\{0,1\}^N$ with the ultra metric ρ.

Let $s \geq 0$. An s-supergale is a function $d : \{0,1\}^* \to [0,\infty)$ such that for each σ:

$$d(\sigma) \geq 2^{-s}[d(\sigma * 0) + d(\sigma * 1)] .$$

d is an s-gale means

$$d(\sigma) = 2^{-s}[d(\sigma * 0) + d(\sigma * 1)]$$

or

$$2^{-s|\sigma|}d(\sigma) = 2^{-s|\sigma*0|}d(\sigma * 0) + 2^{-s|\sigma*1|}d(\sigma * 1)$$

The success set of d is $S^\infty[d] = \{\sigma \in C | \limsup d(\sigma|n) = \infty\}$.

T.: (J. Lutz) Let $E \subset C$. $dim_H(E) = \inf\{s : \exists\ s - gale\ g,\ E \subset S^\infty[d]\}$.

REMARK. One can use this result to define new notions of dimension- constructive dimension or computable dimension of sets by restricting the gales to be constructible or computable. In this way one can obtain the dimension of a point x- the constructive dimension of the singleton set $\{x\}$.

For the random sets we are focusing on we use a generalization of this result and a particular generalized η-gale which allows us to obtain not only the Hausdorff dimension but also the exact dimension function and many other properties. Theses notions were used in [MW, 1986] and [GMW, 1990] to obtain exact dimension in $R^{\mathbb{N}}$.

Let $\{A_1, A_2, \ldots, A_N\}$ be a collection of finite prefix-free subsets of $\{0,1\}^*$, (Think of trees) and let P be a probability distribution on $\{1,\ldots,N\}$. Let $V = \bigcup_{i=1}^{n} A_i$ and enumerate $V = \{v_0, v_1, \ldots, v_M\}$. For $\omega \in \Omega = \{1,\ldots,N\}^{\mathbb{N}}$, define

$$C_0(\omega) = \{0,1\}^{\mathbb{N}},\ C_1(\omega) = \bigcup_{a \in A_{\omega(1)}} [a],$$

and if $C_k(\omega)$ has been defined and is $[\tau_{k,1}] \cup [\tau_{k,2}] \cup \cdots \cup [\tau_{k,n_k}]$, and the last used letter of ω is $\omega(N_k)$, then define

$$C_{k+1}(\omega) = \bigcup_{\substack{1 \le i \le n_k, \\ \sigma \in \omega(N_k + i)}} [\tau_{k,i} * \sigma].$$

Finally, define

$$K(\omega) = \bigcap_{k \ge 1} C_k(\omega).$$

When ω is random according to P, we call $K(\omega)$ a random closed subset of $\{0,1\}^{\mathbb{N}}$.
Again, P also denotes infinite product measure on Ω.

For $\sigma \in \{0, \ldots, M\}^*$, define

$$T_\sigma(\omega) = \begin{cases} 2^{-|v_{\sigma(|\sigma|)}|} & \text{if } [v_{\sigma(1)} * \cdots * v_{\sigma(|\sigma|)}] \cap K(\omega) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Define $J_\sigma(\omega) = [v_{\sigma(1)} * \cdots * v_{\sigma(|\sigma|)}]$ if this cylinder has nonempty intersection with $K(\omega)$ and otherwise is $\emptyset$. Define

$$\ell_\sigma(\omega) = diam J_\sigma(\omega) = \prod_{i=1}^{|\omega|} T_{\sigma|i}(\omega).$$

Notice by conditioning on the tree,

$$E_P[\sum_{i=0}^{M} T_i^\eta] = \sum_{k=1}^{N} P(k) E[\sum_{i=0}^{M} T_i^\eta | \omega(1) = k] = \sum_{i=1}^{N} P(k) \sum_{v \in A_k} 2^{-|v|\eta}.$$

Recall η is the unique number so that

$$E_P[\sum_{i=0}^{M} T_i^\eta] = 1.$$

Let

$$S_n(\omega) = \sum_{|\sigma|=n} \ell_\sigma^\eta(\omega).$$

S_n is our best estimate for the $\mathcal{H}^\eta$ measure of $K(\omega)$, given the first n stages of the construction. Then

$$E[S_{n+1}|\mathcal{F}_n] = E[\sum_{|\sigma|=n} \ell_\sigma^\eta \sum_{p=1}^{M} T_{\sigma*p}^\eta|\mathcal{F}_n] = \sum_{|\sigma|=n} \ell_\sigma^\eta E[\sum_{p=1}^{M} T_{\sigma*p}^\eta|\mathcal{F}_n] = S_n.$$

Thus, $(S_n)_{n\in\mathbb{N}}$ is a nonnegative martingale and converges to a random variable X.

T.: (Mauldin-Williams (1986)) $E[X] = 1$, X has finite moments of all orders and $\mathcal{H}^\eta(K(\omega)) \le X(\omega) < \infty$, for P a.e. ω and $dim_H(K) = \eta$ a.s.

We also define

$$X_\sigma(\omega) = \lim_{n\to\infty} \sum_{|\tau|=n} \ell_{\sigma*\tau}^\eta(\omega).$$

T.: (Mauldin-Williams) For each n, the random variables $(X_\sigma)_{|\sigma|=n}$ are i.i.d. and as a family is independent of $\mathcal{F}_n$ and have the same distribution as $X_\emptyset = X$. Moreover, for each σ,

$$\ell_\sigma^\eta X_\sigma = \sum_{i=0}^{M} \ell_{\sigma*i}^\eta X_{\sigma*i}$$

.

This is the generalized "η-gale". These equations allow us to define a **random measure** μ_ω supported on $K(\omega)$. Define

$$\mu_\omega([\sigma]) = \ell_\sigma^\eta X_\sigma.$$

By Kolmogorov's consistency theorem this extends to a measure on $K(\omega)$ and

$$\mu_\omega(K(\omega)) = X(\omega)$$

. Much information about $K(\omega)$ can be gained from the random measure μ_ω and the mixture measure Q defined next.

I want to indicate how we can use this formalism to:

- First determine the rate at which the diameters $\ell_{\sigma|n}$ are going to zero.

- Second obtain a slowly varying function L so that infinitely often $X_{\sigma|n} \sim L(\ell_{\sigma|n})$

This should allow us with some additional work to conclude $0 < \mathcal{H}^g(K) < \infty$.

Let $D = \{0, ..., M\}^{\mathbb{N}}$. On the product space $D \times \Omega$ we define a probability measure Q as follows.

$$Q(B) = \int \mu_\omega(B_\omega) dP(\omega)$$

We consider the random variables T_k on $D \times \Omega$ defined by $T_k(\sigma, \omega) = T_{\sigma|n}(\omega)$. $(T_k)_{k \in N}$ is an i.i.d. sequence.

Also, we can convert Q expected values into P expected values for certain random variables: If $k \in N$ and $Y : D \times \Omega \mapsto R$ is a random variable with the property that $Y(\sigma, \omega) = y(\sigma', \omega)$ provided $\sigma|k = \sigma'|k$, then

$$E_Q[Y] = E[\sum_{|\sigma|=k} \ell_\sigma^\eta X_\sigma Y(\sigma, \cdot)].$$

In particular,

$$E_Q[|\log T|] = E[\sum_i T_i^\eta |\log T_i|] < \infty$$

As Hausdorff did in his original paper, if one finds $\mathcal{H}^{\eta}(K) = 0$, then one seaches for a slowly varying function L such that $0 < \mathcal{H}^{g}(K) < \infty$, where $g(t) = t^{\eta}L(t)$. The first natural family to try are functions of the form $L(t) = (\log(t))^{\theta}$. We shall see this is too large.

First, note that by the strong law of large numbers

$$\frac{1}{n}|\log \ell_{\sigma|n}| \to B = E_Q[|\log T_1|] = E[\sum T_i^{\eta}|\log T_i|],$$

Q a. s or for P a.e. ω and μ_{ω} a.e. σ. Thus, heuristically,

$$|\log \ell_{\sigma|n}|^{\theta} \sim B^{\theta} n^{\theta}$$

.

Now, for all $t \geq 0$, by Chebyshev's inequality

$$\sum_{n=1}^{\infty} Q(X_n \geq cn^{\theta}) \leq \sum_{n=1}^{\infty} \frac{1}{c^t n^{t\theta}} E_Q[X^t]. \leq \sum_{n=1}^{\infty} \frac{1}{c^t n^{t\theta}} E[X^{t+1}].$$

But, the right hand side is summable for large t, since X has moments of all orders.

By the Borel-Cantelli lemma, this implies that

$$lim_{n\to\infty}\frac{X_{\sigma|n}}{|\log \ell_{\sigma|n}|^{\theta}} = 0,$$

for P a.e. ω and μ_ω a.e. σ. So, $L(t) = |\log t|^{\theta}$ does not work.

T.: (Borel-Cantelli) If $\sum_n Q(A_n) < \infty$ then for Q-a.e. (σ, ω) only finitely many of the A_n's occur.

The second natural scale is $L(t) = (|\log|\log t||)^{\theta}$. Proceeding as before, we have for $c > 0,\ t \geq 0$,

$$(***) \qquad \sum_{n=1}^{\infty} Q(\{(\sigma,\omega) : c^{\theta} X_{\sigma|n}(\omega) \geq (\log n)^{\theta}\}) \leq$$

$$\sum_{n=1}^{\infty} \frac{1}{n^t} E_Q[e^{tcX^{1/\theta}}] = \sum_{n=1}^{\infty} \frac{1}{n^t} E[Xe^{tcX^{1/\theta}}].$$

So, if the moment generating function of $X^{1/\theta}$, has a finite radius of convergence, then (***) is no obstacle to having

$$\limsup_{n\to\infty} \frac{X_{\sigma|n}}{|\log|\log \ell_{\sigma|n}||^{\theta}} > 0,$$

What we finally show is that the moment generating function of $X^{1/\theta}$ has a finite positive radius of convergence , where $\theta = 1 - \eta \log_3 2$. We use this to show that for P a.e. ω there are postive constants m_ω, M_ω such that for μ_ω a.e. σ:

$$m_\omega \leq \limsup \frac{\mu_\omega(B(\sigma,\epsilon))}{g(2\epsilon)} \leq M_\omega$$

It follows from this that for P a.e. ω

$$0 < \mathcal{H}^g(K_\omega) < \infty.$$

Here is a problem that Jack Lutz, Alex McLinden and I considered:

Is it true that there is a number D such that for P a.e. ω and for μ_ω a.e. x

$$dim\ x = cdim\ \{x\} = D?$$

One might conjecture what D is. Let a is the a priori probability distribution on the nodes that that node appears in a tree. If $L_1(\sigma, \omega)$ is the average length of the leaves of the first tree determined by ω, then for every P-random ω, for every μ_ω-random x,

$$dim\ x = \frac{h(a)}{E_Q(L_1)}$$

$h(a)$ is the entropy of a and $E_Q(L_1)$ is a sort of Lyaponov exponent.

Problem, Can one obtain a random multifractal spectrum for the pointwise dimension function?

Thank you.

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