Conformal iterated function systems with overlaps

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1. Introduction

1. $\{S_i\}_{i=1}^N$ is an IFS of injective C^1 conformal contractions on a compact subset $X \subset \mathbb{R}^d$.

2. Each S_i can be extended to a C^1 injective conformal contraction on some open connected $V \supseteq X$ satisfying

$$0 < \inf_{x\in V} \|S_i'(x)\| \leq \sup_{x\in V} \|S_i'(x)\| < 1, \quad 1 \leq i \leq N.$$

3. Let $K \subseteq X$ be the *self-conformal set*:

$$K = \bigcup_{i=1}^N S_i(K),$$

4. For any set of probability weights $\{p_i\}_{i=1}^N$, let μ be the *self-conformal measure*:

$$\mu = \sum_{i=1}^N p_i \mu \circ S_i^{-1}.$$

5. **Problems:** Absolute continuity of μ and $\dim_{\mathrm{H}}(K)$ in the absence of the *open set condition (OSC)*.

Notation and conditions:

$$\Sigma^n = \{1, \ldots, N\}^n, \qquad \Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n.$$

For $I = (i_1, \ldots, i_n) \in \Sigma^n$,

$$S_I := S_{i_1} \circ \cdots \circ S_{i_n}, \;\; r_I := \inf_{x \in V} \|S_I'(x)\|, \;\; R_I := \sup_{x \in V} \|S_I'(x)\|.$$

Main assumptions: Bounded distortion property, weak separation condition.

Bounded distortion property (BDP): $\exists$ constant $C > 0$ such that $\forall\, I \in \Sigma^*$,

$$\frac{\|S_I'(x)\|}{\|S_I'(y)\|} \le C, \quad \forall\, x, y \in V.$$

In particular,

$$r_I \le R_I \le C r_I, \quad \forall\, I \in \Sigma^*.$$

The BDP is satisfied if, say, for each i, $\ln \|S_i'(x)\|$ is Hölder continuous.

For $0 < b < 1$, let

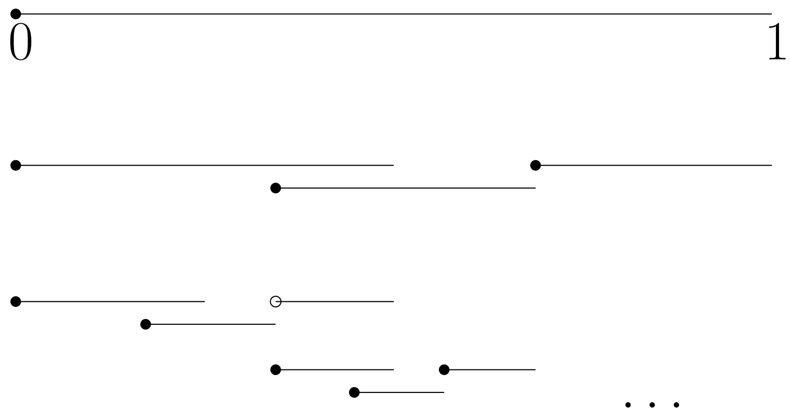
$$\mathcal{I}_b := \{I = (i_1, \ldots, i_n) \in \Sigma^* : \ R_I \le b < R_{i_1,\ldots,i_{n-1}}\},$$
$$\mathcal{A}_b := \{S_I : I \in \mathcal{I}_b\}.$$

Weak separation condition (WSC): $\exists$ constant $\gamma \in \mathbb{N}$ and $D \subseteq X$, $D^\circ \neq \emptyset$, s.t. $\forall\, 0 < b < 1$ and $x \in X$,

$$\#\{S \in \mathcal{A}_b : x \in S(D)\} \le \gamma.$$

Example 1. An IFS satisfying BDP and WSC, with $K = [0, 1]$.

$$S_1(x) = \frac{x}{2}, \ \ S_2(x) = \frac{x^2}{16} + \frac{9x}{32} + \frac{11}{32}, \ \ S_3(x) = \frac{x^2}{32} + \frac{9x}{32} + \frac{11}{16}$$



2. Absolute continuity of self-conformal measure

Theorem 2.1. *(Lau-Wang-N., 2009) Assume the BDP and the WSC. Then an associated self-conformal measure μ is singular with respect to $\mathcal{H}^\alpha|_K$ if and only if there exist $0 < b \leq 1$ and $S \in \mathcal{A}_b$ such that $p_S > R_S^\alpha$, where $p_S := \sum\{p_I : S_I = S,\ I \in \mathcal{I}_b\}$ and $R_S = R_I$ if $S = S_I$.*

For the IFS in Example 1, for any $\{p_i\}$, μ is singular with respect to Lebesgue measure.

3. Hausdorff dimension of self-conformal sets

Theorem 3.1. *(Lau-Wang-N., 2009) Assume the BDP and the WSC. Then*

(a) $\alpha := \dim_{\mathrm{H}}(K) = \dim_{\mathrm{B}}(K)$*;*
(b) $0 < \mathcal{H}^{\alpha}(K) < \infty$*.*

Topological pressure function:

$$P(s) := \lim_{n\to\infty} \frac{1}{n} \ln \sum_{J\in\Sigma^n} R_J^s.$$

P is strictly decreasing, convex, and continuous on $\mathbb{R}$.

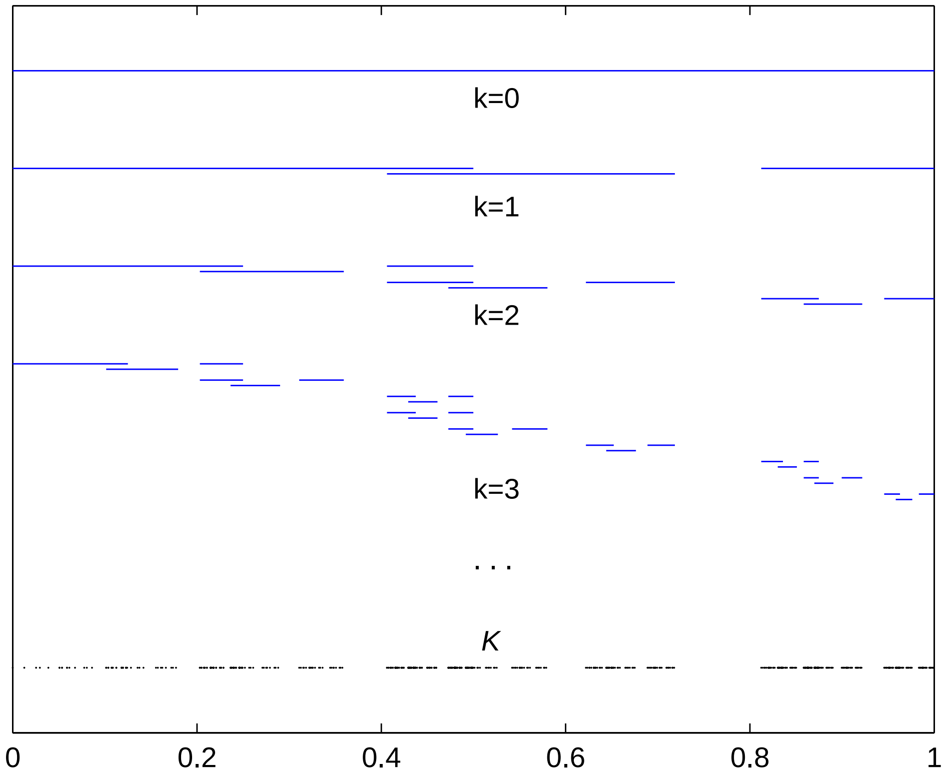
Theorem 3.2. *(Special case of Mauldin-Urbański, 1995) Assume the BDP and OSC. Then* $\dim_{\mathrm{H}}(K) = \alpha$*, where* α *is the unique zero of* P*.*

Theorem 3.3. *(Peres-Rams-Simon-Solomyak, 2001)*

$$OSC \quad \Leftrightarrow \quad 0 < \mathcal{H}^{\alpha}(K) < \infty \text{ and } P(\alpha) = 0.$$

Example 2. A conformal IFS with overlaps but satisfying the WSC:

$$S_1(x) = \frac{x}{2}, \quad S_2(x) = \frac{x^2}{4} + \frac{x}{16} + \frac{13}{32}, \quad S_3(x) = \frac{x^2}{8} + \frac{x}{16} + \frac{13}{16}$$



Modified topological pressure function:

$$\widetilde{P}(s) = \lim_{n\to\infty} \frac{1}{n} \ln \sum_{\phi \in \{S_J : J \in \Sigma^n\}} R_\phi^s.$$

Theorem 3.4. *(Deng-N., 09) Assume the BDP and the WSC. Then* $\dim_{\mathrm{H}}(K)$ *is the unique zero of* $\widetilde{P}$.

Remark: Ferrari has a similar result but his weak separation condition is different and is formulated in terms of the attractor K.

Outline of Proof of Theorem 3.4

(1) Introduce auxiliary topological pressure functions. For $\lambda \in (0,1)$, define

$$\underline{Q}_\lambda(s) = \varliminf_{n\to\infty} \frac{1}{n} \ln \sum_{\phi\in\mathcal{A}_{\lambda^n}} R_\phi^s,$$
$$\overline{Q}_\lambda(s) = \varlimsup_{n\to\infty} \frac{1}{n} \ln \sum_{\phi\in\mathcal{A}_{\lambda^n}} R_\phi^s.$$

Both $\underline{Q}_\lambda(s)$ and $\overline{Q}_\lambda(s)$ are strictly decreasing, continuous on $\mathbb{R}$. $\overline{Q}_\lambda(s)$ is convex.

(2)

Theorem 3.5. *Assume the BDP and the WSC. Then* $\dim_{\mathrm{H}}(K) = \alpha$, *where* α *is the unique zero of* $\underline{Q}_\lambda$ *and* $\overline{Q}_\lambda$.

Upper bound follows from definition of Hausdorff dimension by taking the cover $\{\phi(K) : \phi \in \mathcal{A}_{\lambda_n}\}$.

Lower bound follows by combining WSC and the result $0 < \mathcal{H}^\alpha(K) < \infty$.

(3) Define

$$\mathcal{B}_n := \{S_J : J \in \Sigma^* : r^{n+1} < R_J \leq r^n\} \supseteq \mathcal{A}_{r^n}.$$

Then $\exists$ fixed integer $t > 0$ s.t.

$$\mathcal{B}_n \subseteq \left\{\phi \circ S_I : \ \phi \in \mathcal{A}_{r^n}, \ I \in \bigcup_{i=0}^{t} \Sigma^i\right\}.$$

Hence $\exists$ a constant $C > 0$ s.t.

$$\sum_{\phi \in \mathcal{A}_{r^n}} R_\phi^\alpha \leq \sum_{\phi \in \mathcal{B}_n} R_\phi^\alpha \leq C \sum_{\phi \in \mathcal{A}_{r^n}} R_\phi^\alpha$$

$$\Rightarrow \quad \lim_{n \to \infty} \frac{1}{n} \sum_{\phi \in \mathcal{B}_n} R_\phi^\alpha = \lim_{n \to \infty} \frac{1}{n} \sum_{\phi \in \mathcal{A}_{r^n}} R_\phi^\alpha.$$

(4) For each integer $j \geq 0$, let

$$\begin{aligned}\mathcal{B}_{n,j} &:= \{S_J : J \in \Sigma^n\} \cap \mathcal{B}_j \\ &= \{S_J : J \in \Sigma^n : r^{j+1} < R_J \leq r^j\} \supseteq \mathcal{B}_j.\end{aligned}$$

Then

$$\{S_J : J \in \Sigma^n\} = \bigcup_{j=0}^{n} \{\phi : \phi \in \mathcal{B}_{n,j}\} \subseteq \bigcup_{j=0}^{n} \mathcal{B}_j.$$

Hence,

$$\frac{1}{n} \ln \sum_{\phi \in \{S_J : J \in \Sigma^n\}} R_\phi^\alpha \leq \frac{1}{n} \ln \sum_{j=0}^{n} \sum_{\phi \in \mathcal{B}_{n,j}} R_\phi^\alpha \leq \frac{1}{n} \ln \sum_{j=0}^{n} \sum_{\phi \in \mathcal{B}_j} R_\phi^\alpha$$
$$\to 0$$

Thus,

$$P(\alpha) \leq 0.$$

The proof for $P(\alpha) \geq 0$ is more straightforward.

For $b \in (0,1)$, let α_b be the unique nonnegative number satisfying

$$\sum_{\phi \in \mathcal{A}_b} R_\phi^{\alpha_b} = 1. \tag{3.1}$$

We can show that if $\{S_i\}_{i=1}^N$ has the BDP, then

$$\lim_{b \to 0^+} \frac{\ln \#\mathcal{A}_b}{-\ln b} = \lim_{b \to 0^+} \alpha_b$$

Following Zerner, we call the common value in the *growth dimension* of the IFS $\{S_i\}_{i=1}^N$, and denote it by d_G.

Theorem 3.6. *Assume the BDP. Then*

(a) $\dim_{\mathrm{H}}(K) \leq d_G$.
(b) *If WSC also holds, then* $\dim_{\mathrm{H}}(K) = d_G$.

Theorem 3.7. *Assume that* $\{S_i\}_{i=1}^N$ *satisfies the BDP. Then the OSC is satisfied if and only if the WSC is satisfied and* $S_J \neq S_I$ *for all distinct* $I, J \in \Sigma^*$.