Tube formulas and self-similar tilings

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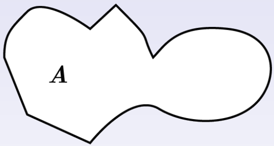
Inner and outer tube formulas

Definition: The *outer ε-neighbourhood* (ε-nbd) of a bounded open set $A \subseteq \mathbb{R}^d$ is

$$A_\varepsilon := \{x \in \overline{A^{\complement}} : dist(x, \mathrm{bd}\, A) \leq \varepsilon\}.$$



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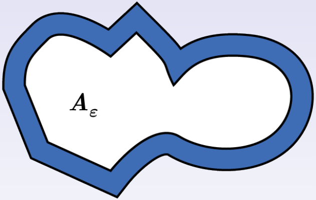
An *outer tube formula* for $A \subseteq \mathbb{R}^d$ is an explicit formula for $\mathrm{vol}_d(A_\varepsilon)$.



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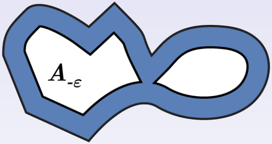


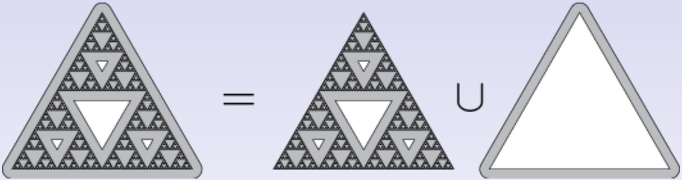
Definition: The *inner ε-nbd* of a bounded open set $A \subseteq \mathbb{R}^d$ is

$$A_{-\varepsilon} := \{x \in \overline{A} : dist(x, A^{\complement}) \leq \varepsilon\} = (A^{\complement})_\varepsilon.$$

An *inner tube formula* is an explicit formula for $\mathrm{vol}_d(A_{-\varepsilon})$.

How to compute the tube formula for a self-similar set?

Use inner tube formula for $A^{\complement}$ to find outer tube formula for A.



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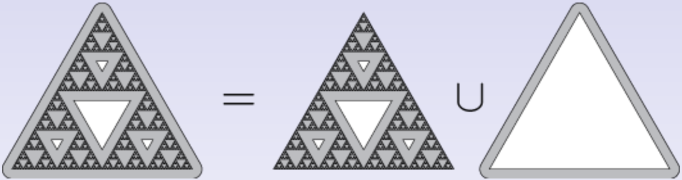

(1) Obtain the components of $A^{\complement}$ from the IFS.

(2) Determine compatibility conditions.

(3) Compute $V(\varepsilon) = \mathrm{vol}_d((A^{\complement})_{-\varepsilon})$ using complex dimensions.

Self-similar sets in $\mathbb{R}^d$

Definition

$\Phi = \{\Phi_1, \ldots, \Phi_J\}$ is a *self-similar system* in $\mathbb{R}^d$ iff

$$\Phi_j(x) = r_j M_j x + t_j, \qquad j = 1, 2, \ldots, J$$

where $0 < r_j < 1$, $M_j \in O(d)$, and $t_j \in \mathbb{R}^d$, for each j.

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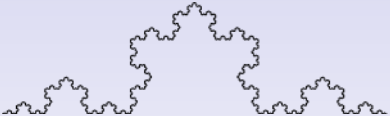
A *self-similar set* $F \subseteq \mathbb{R}^d$ is a fixed point of Φ

$$F = \Phi(F) := \bigcup_{j=1}^{J} \Phi_j(F).$$

The Koch tiling and the Sierpinski gasket tiling

$\Phi_1(z) = \xi\bar{z},\ \Phi_2(z) = (1-\xi)(\bar{z}-1)+1,$ for $\xi = \frac{1}{2} + \frac{\mathbb{i}}{2\sqrt{3}} \in \mathbb{C}$.



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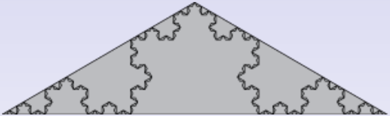

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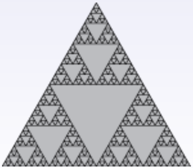

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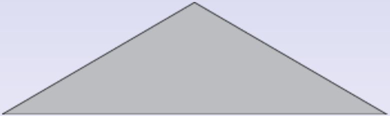

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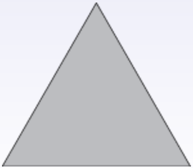

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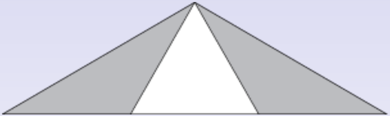

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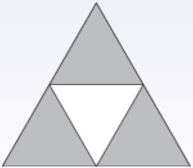

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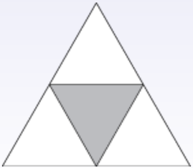

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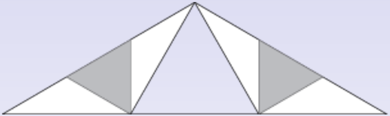

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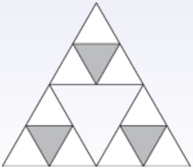

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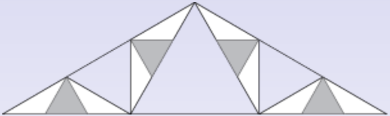

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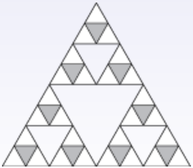

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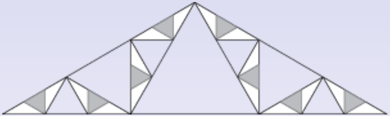

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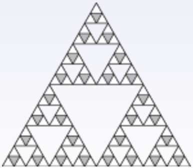

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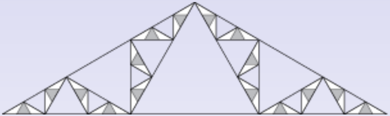

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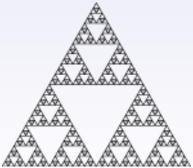

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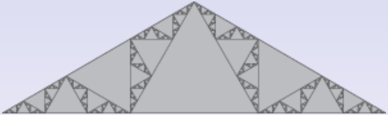

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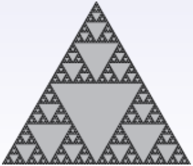

Tiling by open sets

Definition: Let $\mathcal{A} = \{A^i\}_{i \in \mathbb{N}}$ where $A^i \subseteq \mathbb{R}^d$ are disjoint open sets. $\mathcal{A}$ is an *open tiling* of a compact set $K \subseteq \mathbb{R}^d$ iff $K = \overline{\bigcup_{i=1}^{\infty} A^i}$.

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Not a typical tiling:

- Only some compact set $K \supseteq F$ is tiled, not $\mathbb{R}^d$.
- Tiles occur at all scales.
  (Given $\varepsilon > 0$, there is a tile with diameter less than ε.)
- Tiles are open sets.
- No local finiteness is assumed.

Initiating the tiling construction

For the construction to be possible, assume

- F satisfies the *open set condition*, and
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- F satisfies the *open set condition*, and
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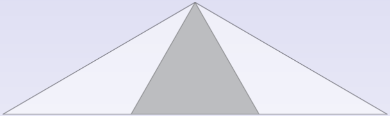
If O is a feasible open set for F, this means:

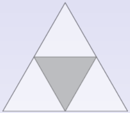
1. $\Phi_j(O) \cap \Phi_k(O) = \varnothing$ for $j \neq k$,
2. $\Phi_j(O) \subseteq O$ for each j,
3. $F \subseteq \overline{O}$, and
4. $O \nsubseteq \Phi(O)$.

First, construct a tiling of $K := \overline{O}$.
Later, worry about which K work for the tube formula.

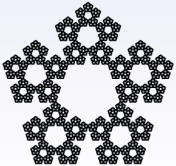
Each tile is the image of a generator

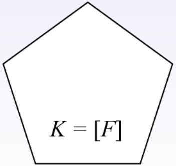
Definition: The *generators* $\{G_q\}_{q=1}^Q$ are the connected components of $\mathrm{int}(K \setminus \Phi(K))$.

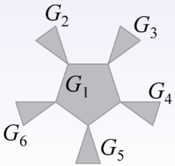




Some examples may have multiple generators.







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Definition: The *self-affine tiling*[1] associated with Φ and O is

$$\mathcal{T} = \mathcal{T}(O) = \{\Phi_w(G_q) \,\vdots\, w \in \mathcal{W}, q = 1, \dots, Q\},$$

where $\mathcal{W} := \bigcup_{k=0}^{\infty}\{1, \dots, N\}^k$ is all finite strings on $\{1, \dots, N\}$, and

$$\Phi_w := \Phi_{w_1} \circ \Phi_{w_2} \circ \dots \circ \Phi_{w_n}.$$

Theorem: $\mathcal{T}(O)$ is an open tiling of $K = \overline{O}$.

Let $T = \bigcup_{R \in \mathcal{T}} R$ denote the union of the tiles.

[1] The tiling construction works for *self-affine sets*, but tube formula technique is only valid for self-similar sets.

How to pick a good O (or K)

Theorem [Compatibility Theorem]: Let $\operatorname{int} F = \varnothing$ satisfy OSC with feasible set O and associated tiling $\mathcal{T}(O)$. Then TFAE:

1. $\operatorname{bd} T = F$.
2. $\operatorname{bd} K \subseteq F$.
3. $\operatorname{bd}(K \setminus \Phi(K)) \subseteq F$.
4. $\operatorname{bd} G_q \subseteq F$ for all $q \in Q$.
5. $F_\varepsilon \cap K = T_{-\varepsilon}$ for all $\varepsilon \geq 0$.
6. $F_\varepsilon \cap K^\complement = K_\varepsilon \cap K^\complement$ for all $\varepsilon \geq 0$.

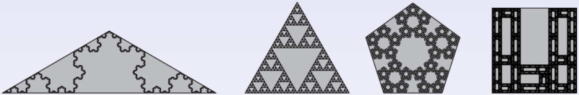
So for a given Φ and F, check that one of 1–4 is satisfied.
Then 5–6 ensure the inner/outer decomposition:

How to pick a good O (or K)

Specific possibilities:

(1) Choose $K = [F]$ and $O = \operatorname{int} K$.

Feasible iff $\operatorname{int} \Phi_j(K) \cap \Phi_k(K) = \varnothing$ for $j \neq k$. (Tileset condition)
In this case, $\operatorname{int} F \neq \varnothing$ iff F is convex. (Nontriviality condition)



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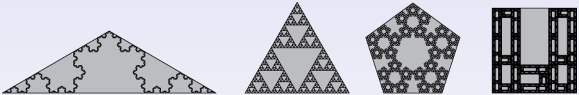

(2) Let U be the unbounded component of $F^{\complement}$.
Choose $K = U^{\complement}$ (the *envelope* of F) and $O = \operatorname{int} K$.

For the envelope, one always has $\operatorname{bd} K \subseteq F \subseteq K \subseteq [F]$,
and K is convex iff $K = [F]$.

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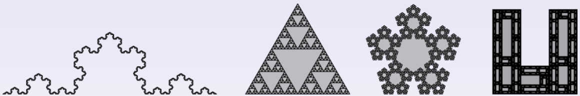

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How to compute the tube formula

Suppose you have

- Φ satisfying OSC, with
- $\operatorname{int} F = \varnothing$, and
- a feasible open set O satisfying the compatibility theorem.

What is the tube formula?

We compute $V(\varepsilon) = \mathrm{vol}_d(T_{-\varepsilon})$, the inner tube formula for the tiling. Wlog, suppose there is only one generator.

The scaling zeta function

Definition: Let $r_w = r_{w_1} r_{w_2} \ldots r_{w_n}$ be the scaling ratio of Φ_w.
The *scaling zeta function* is given by the scaling ratios of Φ via

$$\zeta_{\mathfrak{s}}(s) = \sum_{w \in \mathcal{W}} r_w^s = \frac{1}{1 - \sum_{j=1}^N r_j^s}, \qquad \text{for } s \in \mathbb{C}.$$

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Definition: The *complex dimensions* of Φ are $\mathcal{D}_{\mathfrak{s}} := \{\text{poles of } \zeta_{\mathfrak{s}}\}$.

Theorem: $\dim_{\mathcal{M}}(F) = \sup_{\omega \in \mathcal{D}_{\mathfrak{s}}} \operatorname{Re} \omega = \max\{\omega \in \mathcal{D}_{\mathfrak{s}} : \omega \in \mathbb{R}\}$.

$\zeta_{\mathfrak{s}}$ is the Mellin transform of a measure $\eta_{\mathfrak{s}}$:

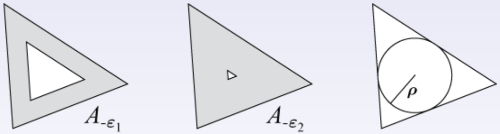
$$\zeta_{\mathfrak{s}}(s) = \int_0^\infty x^{-s} \, \eta_{\mathfrak{s}}(dx), \qquad \eta_{\mathfrak{s}} := \sum_{w \in \mathcal{W}} \delta_{r_w^{-1}}.$$

Converting "scales" to "sizes"

Definition: The *inradius* of $A \subseteq \mathbb{R}^d$ is the radius of the largest metric ball contained in A.

Equivalently, $\rho(A) := \inf\{\varepsilon > 0 : A_{-\varepsilon} = A\}$.

For $A = G$, write $g := \rho(G)$.
The tile $\Phi_w(G)$ has inradius $r_w g$.



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Equivalently, $\rho(A) := \inf\{\varepsilon > 0 : A_{-\varepsilon} = A\}$.

For $A = G$, write $g := \rho(G)$.
The tile $\Phi_w(G)$ has inradius $r_w g$.

Definition: G is *diphase* iff there are constants κ_k so that

$$V_G(\varepsilon) = \sum_{k=0}^{d-1} \kappa_k \varepsilon^{d-k}, \qquad 0 \leq \varepsilon \leq g.$$

Theorem: If A is convex, then $\mathrm{vol}_d(A_\varepsilon) = \sum_{k=0}^{d-1} \kappa_k \varepsilon^{d-k}$ for $\varepsilon \geq 0$.
Here, $\kappa_k = \mu_k(A)\, \mathrm{vol}_{d\text{-}k}(B^{d-k})$, and μ_k are the *intrinsic volumes*.

The tube formula for self-similar tilings

Theorem (Tube formula for simple diphase self-similar tilings)

Suppose $\mathcal{T}$ has a single diphase generator and $\zeta_{\mathfrak{s}}$ has only simple poles. Then for $0 \leq \varepsilon \leq g$,

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathfrak{s}}} \operatorname{res}\left(\zeta_{\mathfrak{s}}; \omega\right) \sum_{k=0}^{d} \frac{g^{\omega-k}}{\omega - k} \kappa_k \varepsilon^{d-\omega} + \sum_{k=0}^{d-1} \kappa_k \zeta_{\mathfrak{s}}(k) \varepsilon^{d-k},$$

where $\mathcal{D}_{\mathfrak{s}} := \{\text{poles of } \zeta_{\mathfrak{s}}\}$.

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$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathfrak{s}} \cup \{0,1,\ldots,d-1\}} c_\omega \varepsilon^{d-\omega}, \qquad \text{for } c_\omega = \operatorname{res}\left(\zeta_{\mathfrak{s}}; \omega\right) \sum_{k=0}^{d} \frac{g^{\omega-k}}{\omega-k} \kappa_k,$$

where $\mathcal{D}_{\mathfrak{s}} := \{\text{poles of } \zeta_{\mathfrak{s}}\}$.

Compare to the Steiner formula:

$$\operatorname{vol}_d(A_\varepsilon) = \sum_{k=0}^{d-1} \kappa_k \varepsilon^{d-k} \text{ for } \varepsilon \geq 0.$$

Geometric zeta function and Steiner-like sets

The *geometric zeta function* of $\mathcal{T}$ with one generator is

$$\zeta_{\mathcal{T}}(\varepsilon, s) := \zeta_{\mathfrak{s}}(s)\varepsilon^{d-s} \sum_{k=0}^{d} \frac{g^{s-k}}{s-k}\kappa_k, \qquad \zeta_{\mathfrak{s}}(s) := \sum_{w \in \mathcal{W}} r_w^s,$$

and it has poles

$$\begin{aligned} \mathcal{D}_{\mathcal{T}} &:= \{\omega : \zeta_{\mathcal{T}}(\varepsilon, s) \text{ has a pole at } s = \omega\} \\ &= \mathcal{D}_{\mathfrak{s}} \cup \{0, 1, \ldots, d-1\}. \end{aligned}$$

These are the complex dimensions of the tiling $\mathcal{T}$.

Geometric zeta function and Steiner-like sets

The *geometric zeta function* of $\mathcal{T}$ with one generator is

$$\zeta_{\mathcal{T}}(\varepsilon, s) := \zeta_{\mathfrak{s}}(s)\varepsilon^{d-s}\sum_{k=0}^{d}\frac{g^{s-k}}{s-k}\kappa_k(G,\varepsilon), \qquad \zeta_{\mathfrak{s}}(s) := \sum_{w\in\mathcal{W}} r_w^s,$$

and it has poles $\mathcal{D}_{\mathcal{T}} := \{0, 1, \ldots, d-1\} \cup \mathcal{D}_{\mathfrak{s}}$.

Definition: A bounded open set G is *Steiner-like* iff

$$V_G(\varepsilon) = \sum_{k=0}^{d}\kappa_k(G,\varepsilon)\varepsilon^{d-k}, \qquad 0 \le \varepsilon \le g,$$

where $\kappa_k(G,\cdot)$ is bounded and locally integrable and

$$\lim_{\varepsilon\to 0^+}\kappa_k(G,\varepsilon)$$

exists, and is positive and finite.

The tube formula for self-similar tilings

Theorem: Suppose $\mathcal{T}$ has a single Steiner-like generator. Then

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{T}}} \operatorname{res}\left(\zeta_{\mathcal{T}}; \omega\right), \qquad \text{for } 0 \leq \varepsilon \leq g.$$

The End

Tube formulas and self-similar tilings

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Fractal strings: the case $d = 1$

Definition

A *fractal string* is simply a bounded open subset $L \subseteq \mathbb{R}$, so L consists of

$$L := \{L_n\}_{n=1}^{\infty},$$

where each L_n is an open interval.

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Essential strategy of fractal strings:

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- ∂L is some fractal set we want to study.

$$\mathcal{L} := \{\ell_n\}_{n=1}^{\infty}, \qquad \sum_{n=1}^{\infty} \ell_n < \infty.$$

$$\ell_1 \geq \ell_2 \geq \ldots > 0.$$

Fractal strings: the case $d = 1$

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where each L_n is an open interval.

When ∂L is a self-similar set, the fractal string L is a self-similar tiling in $\mathbb{R}^d$, $d = 1$.

Fractal strings came first — the self-similar tiling is an extension of this theory to $\mathbb{R}^d$.

$\zeta_{\mathcal{L}}$ relates geometric and arithmetic properties

Definition

The *geometric zeta function* of a fractal string L is

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \ell_n^s = \frac{\sum g_q^s}{1 - \sum r_j^s}, \qquad s \in \mathbb{C}.$$

Theorem: $D = \inf\{\sigma \geq 0 : \sum_{n=1}^{\infty} \ell_n^{\sigma} < \infty\}$.

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Definition

Accordingly, the *complex dimensions* of L are

$$\mathcal{D}_{\mathcal{L}} = \{\omega \in \mathbb{C} : \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}.$$

The (inner) tube formula for fractal strings

Theorem: For a self-similar fractal string L,

$$V(L_{-\varepsilon}) = \sum_{\omega \in \mathcal{D}_{\mathcal{L}}} \frac{(2\varepsilon)^{1-\omega}}{\omega(1-\omega)} \operatorname{res}\left(\zeta_{\mathcal{L}}(s); \omega\right) - 2\varepsilon.$$

(Sum is over the set of complex dimensions.)

The (inner) tube formula for fractal strings

Theorem: For a self-similar fractal string L,

$$V(L_{-\varepsilon}) = \sum_{\omega \in \mathcal{D}_{\mathcal{L}} \cup \{0\}} c_\omega \varepsilon^{1-\omega}.$$

Minkowski content is $\mathcal{M} = \lim_{\varepsilon \to 0+} V(L_{-\varepsilon}) \varepsilon^{-(1-D)}$, when it exists.

- Tube formula shows when string is measurable.

The (inner) tube formula for fractal strings

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Spectral asymptotics: the eigenvalue counting function is

$$N_\nu(x) = x \cdot \mathrm{vol}_1(\mathcal{L}) + \mathrm{res}\,(\zeta_{\mathcal{L}}; D)\, \psi(x) + \mathit{error}, \quad \text{where}$$

$$\psi(x) = \begin{cases} \sum_{n \in \mathbb{Z}} \zeta(D + \mathbb{i} n \mathbf{p}) \frac{x^{D+\mathbb{i}n\mathbf{p}}}{D+\mathbb{i}n\mathbf{p}}, & \text{or} \\ -\zeta(D) \frac{x^D}{D}. \end{cases}$$

Recall: $\mathrm{res}\,(\zeta_{\mathcal{L}}; D)$ is related to D-dimnl volume of ∂L.

$$V(\mathcal{T}_{-\varepsilon}) = \sum_{\omega \in \mathcal{D}_{\mathcal{T}}} \operatorname{res}\left(\zeta_{\mathcal{T}}(\varepsilon, s); \omega\right).$$

$$V(\mathcal{T}_{-\varepsilon}) = \sum_{\omega \in \mathcal{D}_{\mathfrak{s}}} \sum_{k=0}^{d} \operatorname{res}\left(\zeta_{\mathfrak{s}}; \omega\right) g^{\omega-k} \kappa_k \frac{\varepsilon^{d-\omega}}{\omega - k} + \sum_{k=0}^{d-1} g^k \kappa_k \zeta_{\mathfrak{s}}(k) \varepsilon^{d-k}.$$

Compare to strings:

$$V(L_{-\varepsilon}) = \sum_{\omega \in \mathcal{D}_{\mathcal{L}} \cup \{0\}} c_\omega \varepsilon^{1-\omega}.$$

$\mathcal{CS}$ is the complement of the Cantor set in $[0, 1]$

$$\mathcal{CS} = \left\{ \frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \dots \right\},$$

$$\zeta_{\mathcal{CS}}(s) = \sum_{k=0}^{\infty} 2^k 3^{-(k+1)s} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.$$

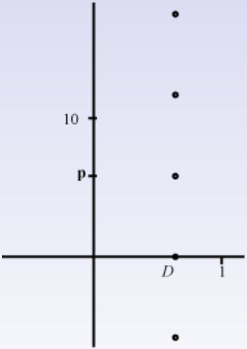
The complex dimensions of $\mathcal{CS}$ (poles of $\zeta_{\mathcal{CS}}$) are

$$\mathcal{D}_{\mathcal{CS}} = \{D + \mathbb{i}n\mathbf{p} : n \in \mathbb{Z}\}, \text{ where}$$

$$D = \log_3 2, \quad \mathbb{i} = \sqrt{-1}, \quad \mathbf{p} = 2\pi / \log 3$$

$\mathcal{CS}$ is an example of a *lattice* string

The complex dimensions $\mathcal{D}_{\mathcal{CS}}$.



Nonlattice example: the Golden String.

Let $r_1 = 2^{-1}$ and $r_2 = 2^{-\phi}$, where $\phi = \frac{1}{2}(1+\sqrt{5})$ is the golden ratio. The Golden String is a *nonlattice*[1] string with lengths

$$\mathcal{GS} = \left\{ \binom{n}{k} \{r_1^k r_2^{n-k}\} : k \leq n = 0, 1, 2, \ldots \right\}$$

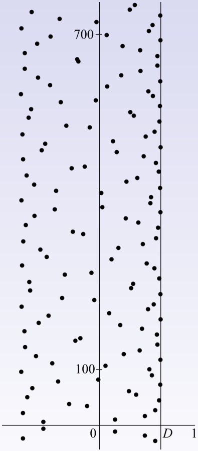
$\binom{n}{k}$ indicates the multiplicity of $r_1^k r_2^{n-k}$.

$$\zeta_{\mathcal{GS}}(s) = \frac{1}{1 - 2^{-s} - 2^{-\phi s}},$$

$\mathcal{D}_{\mathcal{GS}}$ are the solutions of the transcendental equation

$$2^{-\omega} + 2^{-\phi\omega} = 1 \qquad (\omega \in \mathcal{D}),$$

Example 2: the Golden String.



Example 2: the Golden String.

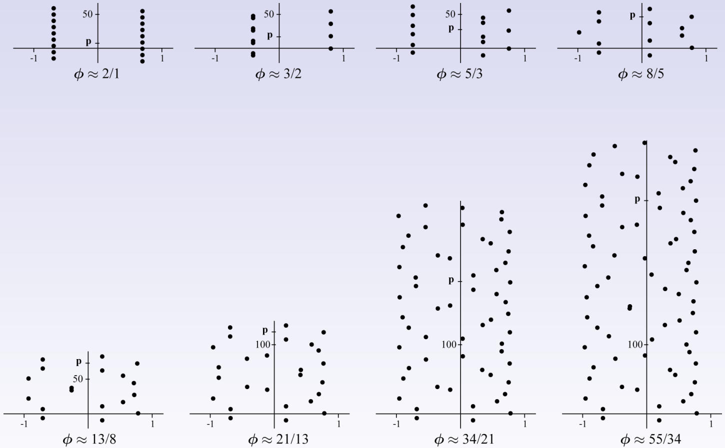
The Golden String is a limit of lattice strings via Diophantine approximation.

$$\frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \ldots \longrightarrow \phi.$$

For any such approximation, r_1 and r_2 are integer powers of some common base r.

$$r_1 = r^{k_1}, \quad r_2 = r^{k_2}, \quad k_1, k_2 \in \mathbb{N}$$

Example 2: the Golden String.



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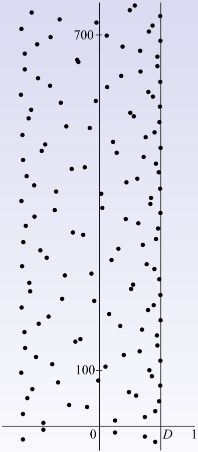


Lattice vs. Nonlattice

For self-similar strings, a dichotomy exists.
The *lattice case*:

- $\{\log r_1, \ldots, \log r_J\}$ are rationally dependent.
- Complex dimns lie on finitely many vert lines.
- There is a row of dimns on $\mathrm{Re}\, s = D$.
- Infinitely many complex dimensions have real part D.
- ∂L is not Minkowski measurable.
- The Cantor String $\mathcal{CS}$ is a lattice string.

Lattice vs. **Nonlattice**

For self-similar strings, a dichotomy exists.
The *nonlattice case*:

- Some $\log r_j$ are rationally independent.
- Complex dimns are scattered in a horizontally bounded strip $[\sigma_l, D]$.
- $\mathrm{Re}\,\omega$ appears dense in the interval $[\sigma_l, D]$.
- D is the only dim with $\mathrm{Re}\,\omega = d$.
- ∂L is Minkowski measurable.
- The golden string $\mathcal{GS}$ is a nonlattice string.

Minkowski dimension is box-counting dimension

Definition

The *Minkowski dimension* of the set $\Omega \subseteq \mathbb{R}^d$

$$D = \dim_M \Omega = \lim_{\varepsilon \to 0^+} \frac{\log M_\varepsilon(\Omega)}{-\log \varepsilon}$$
$$= \inf\{t \geq 0 : V(\Omega_{-\varepsilon}) = O\left(\varepsilon^{d-t}\right) \text{ as } \varepsilon \to 0^+\}.$$

For a string $\Omega = L$, the Minkowski dimension is

$$D = \dim_M \partial L.$$

Minkowski measurability

Definition

The set Ω is *Minkowski measurable* if and only if the limit

$$\mathcal{M} = \mathcal{M}(D;\Omega) = \lim_{\varepsilon\to 0+} V(\Omega_{-\varepsilon})\varepsilon^{-(d-D)}$$

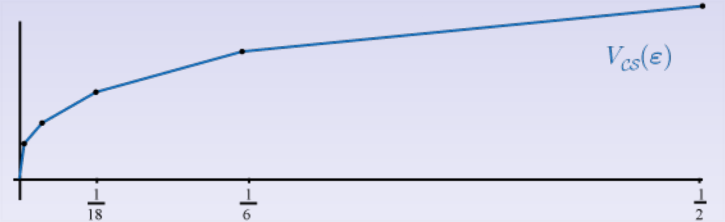
exists, and $0 < \mathcal{M} < \infty$.

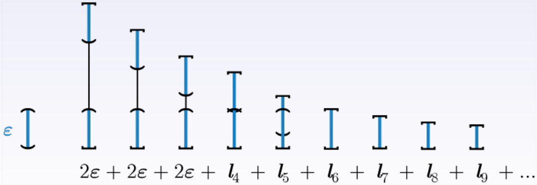
A string $\mathcal{L}$ is measurable iff ∂L is. $\mathcal{M}$ is the *Minkowski content*.

"Measurable" $=$ Minkowski measurable in this talk.

For fractal strings: $V(L_{-\varepsilon}) = \sum_{\ell_n > 2\varepsilon} 2\varepsilon + \sum_{\ell_n \leq 2\varepsilon} \ell_n.$

Using $V(\Omega_{-\varepsilon})$ to see Minkowski measurability





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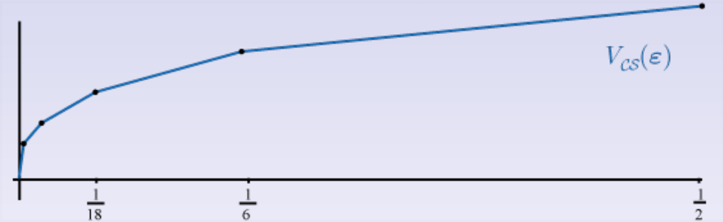

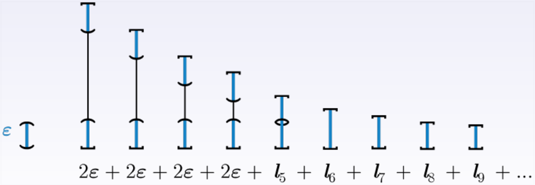

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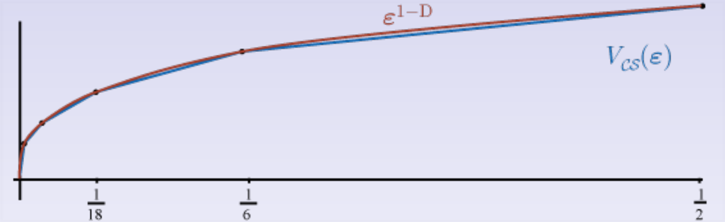

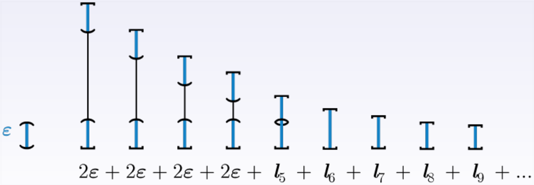

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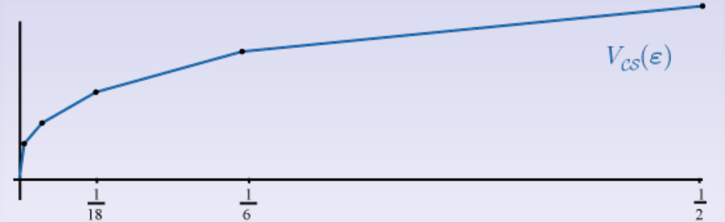

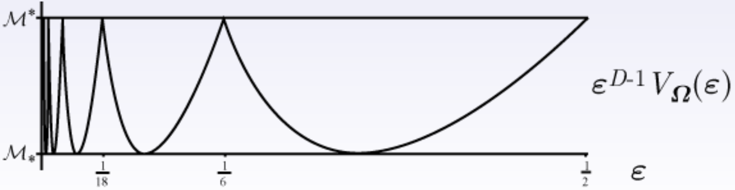

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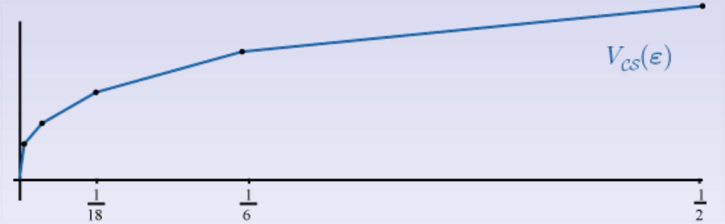

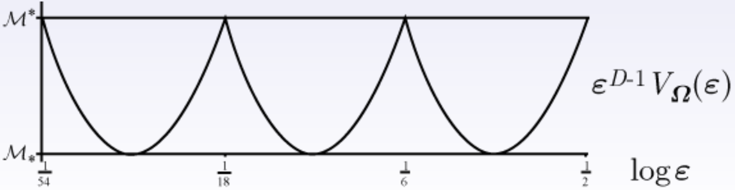

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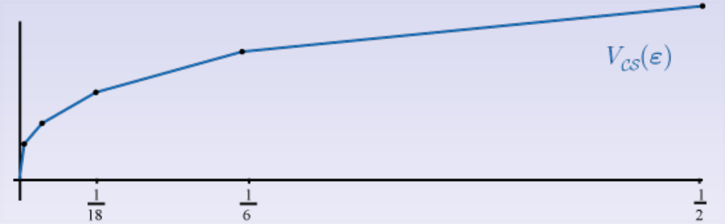

Complex dimensions with real part D induce oscillations in $V(L_{-\varepsilon})$ of order D ("*geometric oscillations*").

This means $\lim_{\varepsilon \to 0^+} V(L_{-\varepsilon})\varepsilon^{-(1-D)}$ cannot exist; it contains terms of the form

$$c_\omega \varepsilon^{\mathrm{Im}(\omega)\mathbb{i}} = c_\omega e^{\mathrm{Im}(\omega)\mathbb{i}\log\varepsilon}.$$