A homology theory for basic sets

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Axiom A (S. Smale,1967): M a compact Riemannian manifold, $\varphi : M \to M$ a diffeomorphism with a hyperbolic structure on the set $X \subset M$ of non-wandering points.

Typically, X is some type of fractal. X may be decomposed into irreducible pieces called basic sets.

Smale space (D. Ruelle): a purely topological version.

(X, d) compact metric space, φ homeomorphism. Each point x has neighbourhoods

$$X^s(x, \epsilon) \times X^u(x, \epsilon),$$

local stable and unstable sets, with

$$\begin{aligned} d(\varphi(y), \varphi(z)) &\leq \lambda d(y, z), \quad y, z \in X^s(x, \epsilon), \\ d(\varphi^{-1}(y), \varphi^{-1}(z)) &\leq \lambda d(y, z), \quad y, z \in X^u(x, \epsilon), \end{aligned}$$

with $0 < \lambda < 1$.

Hyperbolic toral automorphism:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$$

Shifts of finite type:

Let $G = (G^0, G^1, i, t)$ be a finite directed graph. Then

$$\begin{aligned} \Sigma_G &= \{(e^k)_{k=-\infty}^{\infty} \mid e^k \in G^1, \\ &\quad i(e^{k+1}) = t(e^k), \text{ for all } k\} \\ \sigma(e)^k &= e^{k+1}, \text{ "left shift"} \end{aligned}$$

The local product structure is given by

$$\Sigma^s(e, 2^{-n}) = \{(\ldots, *, *, *, e^{-n}, e^{1-n}, e^{2-n} \ldots)\}$$
$$\Sigma^u(e, 2^{-n}) = \{(\ldots, e^{n-2}, e^{n-1}, e^n, *, *, *, \ldots)\}$$

A shift of finite type is any system conjugate to (Σ_G, σ), for some G.

Why shifts of finite type are important.

Theorem 1. *The shifts of finite type are exactly the totally disconnected Smale spaces.*

Theorem 2 (Bowen)**.** *If* (X, φ) *is any irreducible Smale space, then there exists*

$$\pi : (\Sigma, \sigma) \to (X, \varphi),$$

continuous, surjective and finite-to-one.

Proof is by constructing Markov partitions (codings).

Krieger's invariant (to come....)

Solenoid: Start with $z \in \mathbb{T} \to z^m \in \mathbb{T}$; expanding, but not a homeomorphism.

$$\begin{aligned} X &= \lim \mathbb{T} \leftarrow \mathbb{T} \leftarrow \cdots \\ &= \{(z_k)_{k \geq 0} \mid z_{k+1}^m = z_k\}. \\ \varphi(z)_k &= z_k^m. \end{aligned}$$

We have $X^s(x, \epsilon) \sim \text{Cantor}, X^u(x, \epsilon) \sim (-\epsilon, \epsilon)$.

Let G be the graph with one vertex, m edges,so Σ_G is the full m-shift. There is a factor map

$$\pi : (\Sigma_G, \sigma) \to (X, \varphi)$$

based on the base m expansion of real numbers.

This map has a special property: for all e in $\Sigma_G, \epsilon > 0$, $\pi(\Sigma^s(e, \epsilon))$ is *open* in $X^s(\pi(e), \epsilon')$ and

$$\pi : \Sigma^s(e, \epsilon) \to \pi(\Sigma^s(e, \epsilon))$$

is a homeomorphism. We say such a map is *s-bijective*.

Krieger's Invariant (1978 ?)

Krieger gave a construction of a C^*-algebra from a shift of finite type, whose K-theory was readily computed. He also gave a dynamical interpretation.

Let G be a directed graph and let N be the number of vertices in G. Let A be the adjacency matrix for the graph G.

$$D^s(\Sigma_G, \sigma) = \lim \mathbb{Z}^N \xrightarrow{A} \mathbb{Z}^N \xrightarrow{A} \cdots$$

i.e. $\mathbb{Z}^N \times \mathbb{N}/(x, k) \sim (Ax, k+1)$.

- If G has one vertex and m edges, then $N = 1$, $A = [m]$ and $D^s(\Sigma_G, \sigma) \cong \mathbb{Z}[1/m]$.

- If $A = 0$, then $D^s(\Sigma_G, \sigma) \cong 0$.

- If $\det(A) = \pm 1$,then $D^s(\Sigma_G, \sigma) \cong \mathbb{Z}^N$.

D^s as a functor

If $\pi : (\Sigma, \sigma) \to (\Sigma', \sigma)$ is a factor map, does it induce a group homomorphism between the invariants $D^s(\Sigma, \sigma)$ and $D^s(\Sigma', \sigma)$?

If π is s-bijective, then there is a map

$$\pi^s : D^s(\Sigma, \sigma) \to D^s(\Sigma', \sigma).$$

If π is u-bijective, then there is a map

$$\pi^{s*} : D^s(\Sigma', \sigma) \to D^s(\Sigma, \sigma)$$

(Kitchens, Boyle, Marcus, Trow)

Problem: Can we extend Krieger's invariant to an arbitrary a Smale space, (X, φ) ?

Idea: Start with Bowen:

$$\pi : (\Sigma, \sigma) \to (X, \varphi).$$

For $N \geq 0$, define

$$\begin{aligned}\Sigma_N(\pi) \quad = \quad & \{(e_0, e_1, \ldots, e_N) \mid \\ & \pi(e_n) = \pi(e_0), \\ & 0 \leq n \leq N\}.\end{aligned}$$

For all $N \geq 0$, $(\Sigma_N(\pi), \sigma)$ is also a shift of finite type and we can consider $D^s(\Sigma_N(\pi), \sigma)$. We also have $\delta_n : (\Sigma_N(\pi), \sigma) \to (\Sigma_{N-1}(\pi), \sigma)$ which erases e_n, $0 \leq n \leq N$.

Idea: Compute homology of (X, φ) from $D^s(\Sigma_N(\pi), \sigma), N \geq 0$ and the factor maps δ_n from $(\Sigma_N(\pi), \sigma)$ to $(\Sigma_{N-1}(\pi), \sigma)$.

Problem: Functoriality of δ_n.

Homology: Second attempt

Theorem 3 (Better Bowen)**.** *Let* (X, φ) *be an irreducible Smale space. There exists a Smale space* (Y, ψ) *with* $Y^u(y, \epsilon)$ *totally disconnected for all* y *and an* s*-bijective factor map*

$$\pi_s : (Y, \psi) \to (X, \varphi).$$

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That is, $Y^u(y, \epsilon)$ is totally disconnected, while $Y^s(y, \epsilon)$ is homeomorphic to $X^s(\pi_s(y), \epsilon)$.

This is a "one-coordinate" version of Bowen's Theorem.

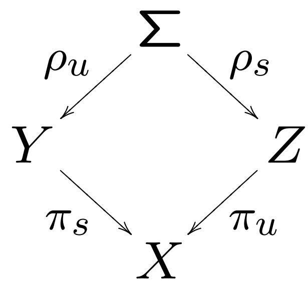
Reversing the rôles of stable and unstable, find (Z, ζ, π_u).

We call $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ an s/u*-bijective pair* for (X, φ).

Consider the fibred product:

$$\Sigma = \{(y,z) \in Y \times Z \mid \pi_s(y) = \pi_u(z)\}$$

with



$\rho_s(y,z) = z$ is s-bijective, $\rho_u(y,z) = y$ is u-bijective. It follows that Σ is a SFT.

For $L, M \geq 0$,

$$\begin{aligned} \Sigma_{L,M}(\pi) = \{ & (y_0, \ldots, y_L, z_0, \ldots, z_M) \mid \\ & y_l \in Y, z_m \in Z, \\ & \pi_s(y_l) = \pi_u(z_m)\}. \end{aligned}$$

is a shift of finite type. Moreover, the maps

$$\delta_{l,} : \Sigma_{L,M} \to \Sigma_{L-1,M}, \delta_{,m} : \Sigma_{L,M} \to \Sigma_{L,M-1}$$

which erase y_l and z_m are s-bijective and u-bijective, respectively.

We get a double complex:

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,2}) & \longleftarrow & D^s(\Sigma_{1,2}) & \longleftarrow & D^s(\Sigma_{2,2}) & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,1}) & \longleftarrow & D^s(\Sigma_{1,1}) & \longleftarrow & D^s(\Sigma_{2,1}) & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,0}) & \longleftarrow & D^s(\Sigma_{1,0}) & \longleftarrow & D^s(\Sigma_{2,0}) & \longleftarrow &
\end{array}$$

$$\begin{array}{rc}
\partial_N^s : & \oplus_{L-M=N} D^s(\Sigma_{L,M}) \\
\rightarrow & \oplus_{L-M=N-1} D^s(\Sigma_{L,M})
\end{array}$$

$$\partial_N^s = \sum_{l=0}^{L}(-1)^l \delta_{l,}^s + \sum_{m=0}^{M+1}(-1)^{m+M}\delta_{,m}^{s*}$$

$$H_N^s(\pi) = \ker(\partial_N^s)/Im(\partial_{N+1}^s).$$

Basic theorems

Recall: beginning with (X, φ), we select an s/u-bijective pair $\pi = (Y, \pi_s, Z, \pi_u)$ and compute $H^s_N(\pi)$.

Theorem 4. *The groups $H^s_N(\pi)$ do not depend on the choice of s/u-bijective pair $\pi = (Y, \pi_s, Z, \pi_u)$, but only on (X, φ).*

From now on, we write $H^s_N(X, \varphi)$.

Theorem 5. *The functor $H^s_*(X, \varphi)$ is covariant for s-bijective maps, contravariant for u-bijective maps.*

Theorem 6. *For all N, $H^s_N(X, \varphi)$ is finite rank and only finitely many are non-zero.*

We can regard $\varphi : (X, \varphi) \to (X, \varphi)$, which is both s and u-bijective and so induces an automorphism of the invariants.

Theorem 7. *(Lefschetz Formula) Let (X, φ) be any Smale space having an s/u-bijective pair and let $p \geq 1$.*

$$\begin{aligned} \sum_{N \in \mathbb{Z}} (-1)^N \quad Tr[(\varphi^s)^p : \quad & H^s_N(X, \varphi) \otimes \mathbb{Q} \\ \to \quad & H^s_N(X, \varphi) \otimes \mathbb{Q}] \\ = \quad & \#\{x \in X \mid \varphi^p(x) = x\} \end{aligned}$$

Example 1: Shifts of finite type

If $(X, \varphi) = (\Sigma, \sigma)$, then $Y = \Sigma = Z$ is an s/u-bijective pair.

N	$H^s_N(\Sigma, \sigma)$
0	$D^s(\Sigma, \sigma)$
$\neq 0$	0

Example 2: $(\mathbf{X}, \varphi) = \mathbf{m}^{\infty}$**-solenoid** (Bazett-P.)

An s/u-bijective pair is $Y = \{0, 1, \ldots, m-1\}^{\mathbb{Z}}$, the full m-shift, $Z = X$. We get

N	$H^s_N(X, \varphi)$
0	$\mathbb{Z}[1/m]$
1	$\mathbb{Z}$
$\neq 0, 1$	0

Example 3: A hyperbolic toral automorphism (Bazett-P.):

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$$

We get

N	$H^s_N(X,\varphi)$	φ^s
-1	$\mathbb{Z}$	1
0	$\mathbb{Z}^2$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
1	$\mathbb{Z}$	-1
$\neq \pm 1, 0$	0	0