Resolvent kernel estimates on P.C.F.S.S. fractals

Luke G. Rogers

PCFSS Set

- Contractions $F_1, \ldots, F_N$ on complete metric space.
- Self-similar set X (usually fractal).

$$X = \bigcup_{1}^{N} F_n(X)$$

- For word $w = w_1 \ldots w_m$, call $F_w = F_{w_1} \circ \cdots F_{w_m}(X)$ an m-cell.
- Post-critically finite if there is finite set V_0 such that cells intersect only at points of sets $F_w(V_0)$, w a word.
- Examples: Unit Interval, Sierpinski Gasket
- Non-example: Sierpinski Carpet

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Key Idea: PCFSS sets can be viewed as limits of graphs.

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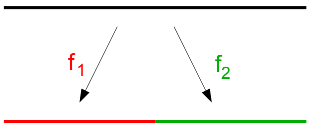

The Unit Interval

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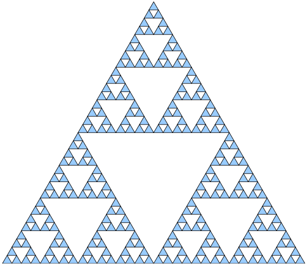

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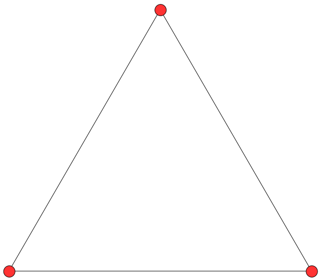

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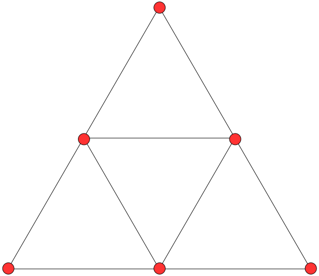

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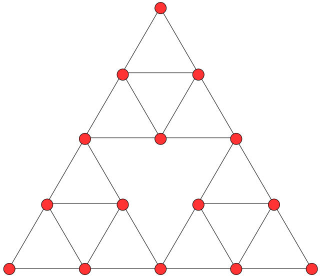

Laplacian

- Basic differential operator is Laplacian Δ
- It is a scaling limit of graph Laplacians

$$\Delta_m u(x) = \sum_{y \sim_m x} \bigl(u(y) - u(x)\bigr) \rightsquigarrow \Delta u(x)$$

- Symbol $\rightsquigarrow$ hides scaling information of two types:
  - μ_w factor corresponding to measure μ on set
  - r_w factor corresponding to energy

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Resolvent kernel for Laplacian

- For z not in spectrum, consider resolvent $(z - \Delta)^{-1}$.
- Look at resolvent kernel: function $G^{(z)}(x, y)$ such that

$$(z - \Delta)^{-1} u(x) = \int_X u(y) G^{(z)}(x, y) \, d\mu(y)$$

- Goal: Understand structure of $G^{(z)}(x, y)$ and obtain estimates.
- Reasons:
  - Operators of Laplacian $f(\Delta)$
  - Heat estimates $e^{t\Delta}$ (Kumagai, Fitzsimmons, Hambly)

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Structure of Resolvent Kernel

- **Theorem [Ionescu, Pearse, R., Ruan, Strichartz]** For suitable z, the resolvent kernel may be written as a self-similar series:

$$G^{(z)}(x,y) = \sum_{w \in W_*} r_w \Psi^{(r_w \mu_w z)}(F_w^{-1}x, F_w^{-1}y)$$

  where Ψ term lives on cell F_w and solves analogous discrete problem.

- Explicit formula for Ψ in terms of "piecewise eigenfunction" $\eta_p^{(z)}$, which satisfies

$$(z-\Delta)\eta_p^{(z)} = 0$$
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Unit Interval Case

- Piecewise eigenfunction is just Sinh function
- Self-similar decomposition into piecewise eigenfunctions with smaller eigenvalues
- Red bumps are multiples of one fixed bump.
- Heights determined by smoothness requirement

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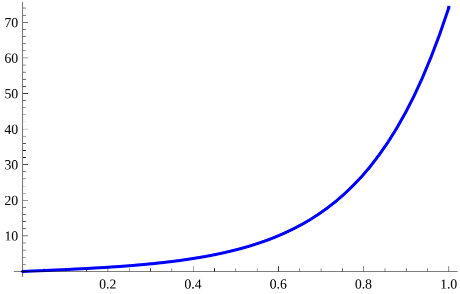

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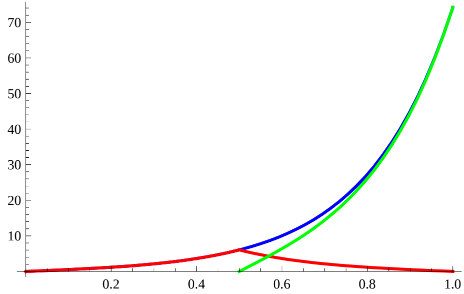

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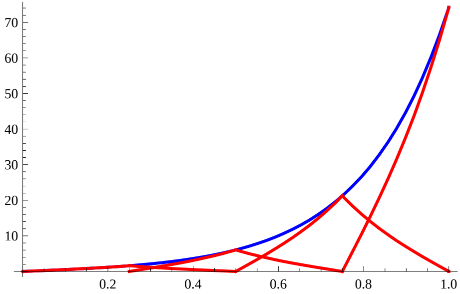

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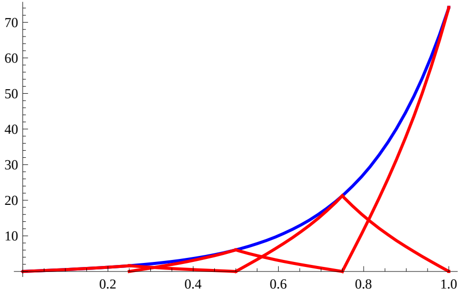

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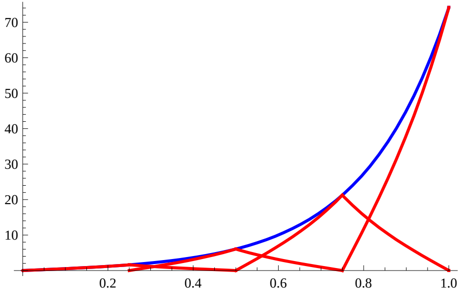

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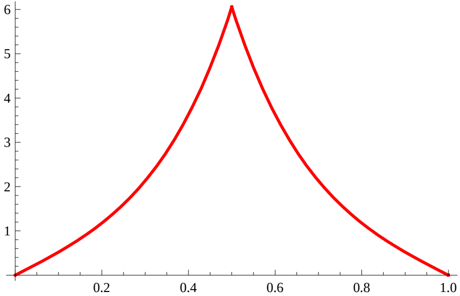
- Slope at top has larger magnitude than slope at bottom



- Smoothness $\Rightarrow$ bump smaller by factor each time
- Function decays exponentially with number of cells
- Number of cells depends on eigenvalue!

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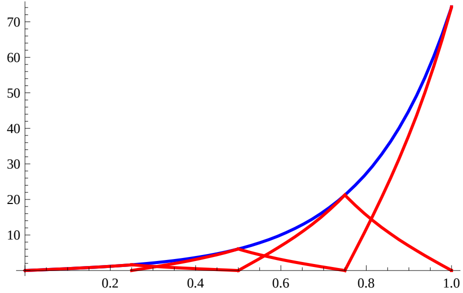

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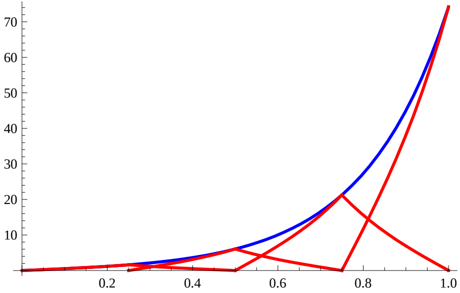

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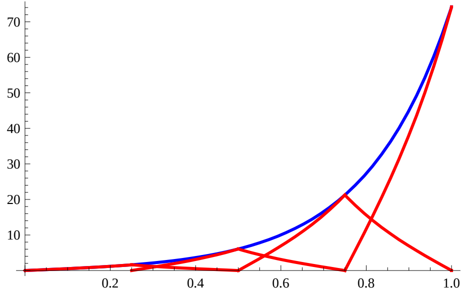

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Decay with Chemical Metric

- For given $z \in (0, \infty)$ decompose fractal so cells have Laplacian scale $\sim z^{-1}$.
- Path distance called "Chemical Metric" $d^{(z)}(x, y)$
- Have showed: Piecewise eigenfunctions decay exponentially with chemical distance
- Some extra work shows off diagonal resolvent kernel decay is similar

$$G^{(z)}(x, y) \leq Cz^{-1/(S+1)} \exp\Bigl(-cd^{(z)}(x, y)\Bigr)$$

- For certain fractals $d^{(z)}(x, y) \approx z^{\gamma}$ some $\gamma \leq \frac{1}{2}$

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Estimates in Sector of $\mathbb{C}$

- Have decay estimate on positive real axis
- Spectral decomposition implies $G^{(z)}$ grows slower than power of $|z|$ in sector
- Multiply $G^{(z)}(x,y)$ by $\exp\bigl(Az^{\gamma}\bigr)$ with $A \in \mathbb{C}$ chosen so Az^{γ} imaginary on ray angle a and real part less than $d^{(z)}(x,y)$ on real axis
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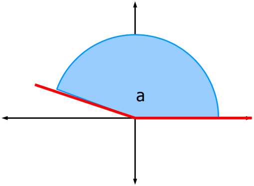

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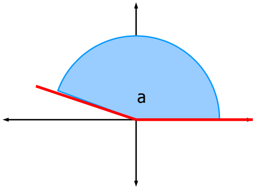

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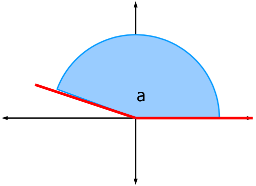

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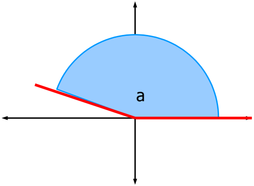

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- Product bounded by Phragmen-Lindelöf theorem
- Hence for some constants

$$\left|G^{(z)}(x,y)\right| \leq C|z|^{-1/(S+1)} \exp\left(-c(x,y)z^{\gamma}\right)$$

Theorem (R.)

The resolvent kernel satisfies

$$\left|G^{(z)}(x,y)\right| \leq C\frac{|z|^{-1/(S+1)}}{\sin \operatorname{Arg}(z)} \exp\left(-c_1 \sin(c_2(\pi - \operatorname{Arg}(z)))d^{(z)}(x,y)\right)$$

- Modified version of Phragmen-Lindelöf deals with cases where chemical metric not like $|z|^{\gamma}$
- Corollary: Upper bounds on heat kernel by contour integration

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- Product bounded by Phragmen-Lindelöf theorem
- Hence for some constants

$$\left|G^{(z)}(x,y)\right| \leq C|z|^{-1/(S+1)} \exp\left(-c(x,y)z^{\gamma}\right)$$

Theorem (R.)

The resolvent kernel satisfies

$$\left|G^{(z)}(x,y)\right| \leq C\frac{|z|^{-1/(S+1)}}{\sin \operatorname{Arg}(z)} \exp\left(-c_1 \sin(c_2(\pi - \operatorname{Arg}(z)))d^{(z)}(x,y)\right)$$

- Modified version of Phragmen-Lindelöf deals with cases where chemical metric not like $|z|^{\gamma}$
- Corollary: Upper bounds on heat kernel by contour integration