Iterated function systems with overlap

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Joint work with

- Palle Jorgensen, University of Iowa
- Keri Kornelson, University of Oklahoma

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Exact measurements of overlap (due to Pengjun Shen)

In the summer of 2008, Pengjun Shen, Grinnell '11, extended the method Keri just described for the golden ratio to numbers λ of the form

$$1 = \lambda + \lambda^2 + \ldots + \lambda^m.$$

For each $m > 1$, $1 = \lambda + \lambda^2 + \ldots + \lambda^m$ has a root λ in $(\frac{1}{2}, 1)$, and as m gets larger, λ approaches $\frac{1}{2}$ from above.

For $m = 3$, $\lambda \approx 0.543869$.
For $m = 10$, $\lambda \approx 0.500245$.
For $m = 15$, $\lambda \approx 0.500008$.

Exact measurements of overlap

Theorem (Pengjun Shen)

If λ is the real root of $1 = \lambda + \lambda^2 + \ldots + \lambda^m$ in $(\frac{1}{2}, 1)$, then

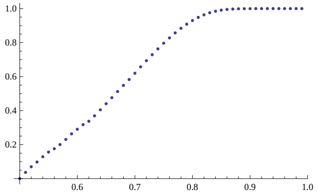
$$\mu_\lambda(\mathcal{O}) = \frac{1}{2^m - 1}.$$

Proof.

Along the same lines as the proof Keri described for $\lambda = \frac{\sqrt{5}-1}{2}$. □

Approximate measures of overlap (Pengjun Shen)

Figure: Values of $\mu_\lambda(\mathcal{O})$, with 50 sample points for $\lambda \in [\frac{1}{2}, 1]$.



Pengjun, with the help of Grinnell computer science faculty member John Stone, wrote a computer program to approximate the measure of the overlap. The program seems to be much more accurate for values of λ near $\frac{1}{2}$ than for values of λ near 1.

Generating Sierpinski gaskets

Let $\mathbf{u}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$, and $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, with $\lambda \in [\frac{1}{2}, 1)$.

We work with the affine contractive IFS given by

$$\tau_0(\mathbf{x}) = A\mathbf{x},$$

$$\tau_1(\mathbf{x}) = A(\mathbf{x} + \mathbf{u}_1),$$

and

$$\tau_2(\mathbf{x}) = A(\mathbf{x} + \mathbf{u}_2).$$

Comparing one and two dimensions

We found a few surprises for the thickened Sierpinski gasket $\mathcal{G}_\lambda$ for $\lambda \in (\frac{1}{2}, 1)$.

The cases are not as simple as in one dimension.

Recall: overlap in one dimension, $\lambda \in [\frac{1}{2}, 1)$, is geometrically simple!

(a) $\lambda = \frac{1}{2}$: the attractor $X_{\frac{1}{2}}$ is an interval, and the overlap $\tau_0(X_{\frac{1}{2}}) \cap \tau_1(X_{\frac{1}{2}})$ is a single point.

(b) $\lambda \in (\frac{1}{2}, 1)$: the attractor X_λ and the overlap $\tau_0(X_\lambda) \cap \tau_1(X_\lambda)$ are both intervals.

Of course, while the actual overlap set is simple, the harmonic analysis for even this case is not simple at all!

Overlap in two dimensions: more subtleties!

In two dimensions, there are 5 cases to consider.
Case 1: the "usual" Sierpinski gasket $\mathcal{G}_{\frac{1}{2}}$.

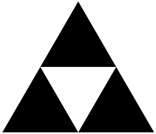


Figure: One iteration. Picture from Keri Kornelson.

- Overlap occurs only at the vertices of the triangles.
- The overlap in $\mathcal{G}_{\frac{1}{2}}$ is a countable set of singleton points.

Overlap at level n

Overlap at level n helps us study the overlap at the n^{th} stage in the generation of the attractor $\mathcal{G}_\lambda$.

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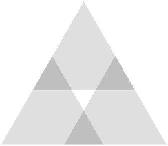


$\tau_0(T) \cup \tau_1(T) \cup \tau_2(T)$

Overlap pictures generated by Brian Treadway on *Mathematica*.

Overlap at level n

Overlap of level n, or $\mathbf{ov}(\tau^n(T))$, refers to overlap of monomials in the τ_is of degree n. For example,

$$\mathbf{ov}(\tau^1(T)) = (\tau_0(T)\cap\tau_1(T))\cup(\tau_0(T)\cap\tau_2(T))\cup(\tau_1(T)\cap\tau_2(T)),$$

and

$$\begin{aligned}
&\mathbf{ov}(\tau^2(T))\\
&= (\tau_0\tau_0(T)\cap\tau_0\tau_1(T))\cup(\tau_0\tau_0(T)\cap\tau_0\tau_2(T))\cup(\tau_0\tau_1(T)\cap\tau_0\tau_2(T))\\
&\cup(\tau_1\tau_0(T)\cap\tau_1\tau_1(T))\cup(\tau_1\tau_0(T)\cap\tau_1\tau_2(T))\cup(\tau_1\tau_1(T)\cap\tau_1\tau_2(T))\\
&\cup(\tau_2\tau_0(T)\cap\tau_2\tau_1(T))\cup(\tau_2\tau_0(T)\cap\tau_2\tau_2(T))\cup(\tau_2\tau_1(T)\cap\tau_2\tau_2(T)).
\end{aligned}$$

Second case: $\lambda \in (\frac{1}{2}, \frac{\sqrt{5}-1}{2})$

In $\mathcal{G}_\lambda$, $\lambda \in (\frac{1}{2}, \frac{\sqrt{5}-1}{2})$,

$$\mathbf{ov}(\tau^n(T)) \cap \mathbf{ov}(\tau^{n+1}(T)) = \emptyset.$$

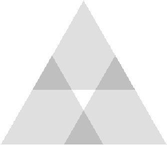
For example, $n = 1$:

$\mathbf{ov}(\tau^1(T))$

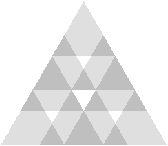
$\mathbf{ov}(\tau^2(T))$:
the smaller shaded triangles

Third case: $\lambda = \frac{\sqrt{5}-1}{2}$

The nature of the overlaps **changes** at the critical point $\lambda = \frac{\sqrt{5}-1}{2}$. For example, we see that $\mathbf{ov}(\tau^1(T))$ and $\mathbf{ov}(\tau^2(T))$ have non-trivial intersection.



$\mathbf{ov}(\tau^1(T))$



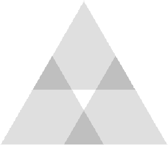
$\mathbf{ov}(\tau^1(T)) \cup \mathbf{ov}(\tau^2(T))$

The small triangles share at least one vertex each with the larger shaded triangles.

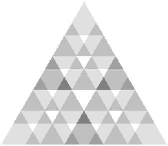
Third case: $\lambda = \frac{\sqrt{5}-1}{2}$

Furthermore, at stage n, one of two things can occur:

- the intersection of the interiors of $\mathbf{ov}(\tau^n(T))$ and triangles of $\mathbf{ov}(\tau^{n+2}(T))$ is empty
- triangles of $\mathbf{ov}(\tau^{n+2}(T))$ are completely contained within $\mathbf{ov}(\tau^n(T))$



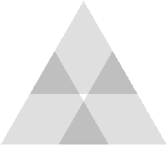
$\mathbf{ov}(\tau^1(T))$



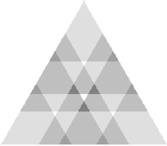
$\bigcup_{n=1}^{3} \mathbf{ov}(\tau^n(T))$

Fourth case: $\lambda \in (\frac{\sqrt{5}-1}{2}, \frac{2}{3})$

For $\lambda \in (\frac{\sqrt{5}-1}{2}, \frac{2}{3})$, $\mathbf{ov}(\tau^n(T)) \cap \mathbf{ov}(\tau^{n+1}(T))$ is uncountable. For example, $n = 1$ again:



$\mathbf{ov}(\tau^1(T))$



$\mathbf{ov}(\tau^1(T)) \cup \mathbf{ov}(\tau^2(T))$

But notice that there are still gaps!

Fifth case: $\lambda \in [\frac{2}{3}, 1)$

Finally, when $\lambda \in [\frac{2}{3}, 1)$, the gaps close, but the overlap still remains.

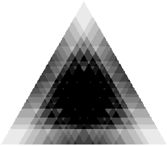


Figure: Four iterations for $\mathcal{G}_{\frac{3}{4}}$. No gaps!

Symmetry

Notation:

- $\Omega = \{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2\}^{\mathbb{N}}$

Symmetry

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- $\Omega = \{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2\}^{\mathbb{N}}$
  Ω is the set of infinite strings

$$\omega = (\mathbf{u}_{i_1}, \mathbf{u}_{i_2}, \ldots, \mathbf{u}_{i_n}, \ldots).$$

Symmetry

Notation:

- $\Omega = \{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2\}^{\mathbb{N}}$
- $P_{\frac{1}{3}}$ = Bernoulli measure on Ω

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- $P_{\frac{1}{3}}$ = Bernoulli measure on Ω
  Define $P_{\frac{1}{3}}$ on cylinders and extend to all of Ω. Cylinders are determined by finitely many elements at the beginning of a word $\omega \in \Omega$. If $w = (w_1, w_2, \ldots, w_n)$ is a finite word, then a cylinder in Ω is

$$\Omega(w) = \{\omega \in \Omega : \omega_1 = w_1, \omega_2 = w_2, \ldots, \omega_n = w_n\}.$$

  Then

$$P_{\frac{1}{3}}(\Omega(w)) = \frac{1}{3^n}.$$

Symmetry

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- $P_{\frac{1}{3}}$ = Bernoulli measure on Ω
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- μ equilibrium (Hutchinson) measure on $\mathcal{G}$ formed with equal weights:

$$\mu = P_{\frac{1}{3}} \circ \pi^{-1} = \frac{1}{3} \sum_{i=0}^{2} \mu \circ \tau_i^{-1}$$

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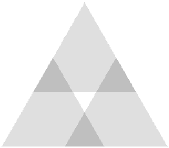


$\tau_0(T) \cup \tau_1(T) \cup \tau_2(T)$

The measure of $\mu(\tau_i(T))$

We can repeat the same argument from Keri's talk, adjusted to two dimensions, to show that

$$P_{\frac{1}{3}}(\{\omega \in \Omega : \pi(\omega) \in \tau_0(T)\}) = P_{\frac{1}{3}}(\{\omega \in \Omega : \pi(\omega) \in \tau_1(T)\})$$

and

$$P_{\frac{1}{3}}(\{\omega \in \Omega : \pi(\omega) \in \tau_0(T)\}) = P_{\frac{1}{3}}(\{\omega \in \Omega : \pi(\omega) \in \tau_2(T)\}).$$

The measure of $\mu(\tau_i(T))$

Therefore,

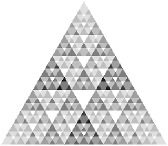
$$\mu(\tau_0(T)) = \mu(\tau_1(T)) = \mu(\tau_2(T)).$$

In other words, we can think of the equilibrium measure being distributed "evenly" over the three pieces which compose the first iteration of the gasket $\mathcal{G}_\lambda$.

Symmetry, continued

Again, let $\mathcal{G}_\lambda$ be the thickened Sierpinski gasket. We define the $(i,j)^{\text{th}}$ overlap $OV_{i,j}$ as

$$OV_{i,j} := \tau_i(\mathcal{G}_\lambda) \cap \tau_j(\mathcal{G}_\lambda).$$



In the special case $\lambda = \frac{\sqrt{5}-1}{2}$, we can modify the one-dimensional argument to show that $\mu(OV_{i,j}) = \frac{1}{24}$ for all pairs (i,j).

Essential overlap

For every $\lambda \in (\frac{1}{2}, 1)$, both X_λ and $\mathcal{G}_\lambda$ have **essential** overlap.

Definition

Let $\{\tau_i\}$ be a contractive IFS with attractor X and equilibrium measure μ. We say that the IFS has **essential overlap** when $\sum_{i \neq j} \mu(\tau_i(X) \cap \tau_j(X)) \neq 0$.

For all $\lambda \in (\frac{1}{2}, 1)$, essential overlap exists for both X_λ (one dimension) and $\mathcal{G}_\lambda$ (two dimensions).

Column isometries

Let $\mathcal{H}$ be a complex Hilbert space, and let $\{F_i : 1 \leq i \leq N\}$ be a set of bounded operators on $\mathcal{H}$.

Definition

We say that $(F_1, F_2, \ldots, F_N)$ is a **column isometry** if the mapping

$$\mathbb{F} : \mathcal{H} \to \begin{pmatrix} \mathcal{H} \\ \oplus \\ \vdots \\ \oplus \\ \mathcal{H} \end{pmatrix} \qquad \text{defined by} \qquad \mathbb{F}(\xi) = \begin{pmatrix} F_1(\xi) \\ \vdots \\ F_N(\xi) \end{pmatrix}$$

is an **isometry** (in general not onto).

The adjoint of $\mathbb{F}$

The adjoint $\mathbb{F}^*$ can be identifed as a row operator:

$$\mathbb{F}^* : \mathcal{H} \oplus \ldots \oplus \mathcal{H} \to \mathcal{H}$$

is given by

$$\mathbb{F}^*(\xi_1, \ldots, \xi_N) = \sum_{i=1}^{N} F_i^* \xi_i.$$

Significance of the adjoint $\mathbb{F}^*$: The column isometry $\mathbb{F}$ is onto if and only if $\mathbb{F}\mathbb{F}^*$ is the identity on the direct sum $\begin{pmatrix} \mathcal{H} \\ \oplus \\ \vdots \\ \oplus \\ \mathcal{H} \end{pmatrix}$.

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Significance of the adjoint $\mathbb{F}^*$: The column isometry $\mathbb{F}$ is onto if and only if $\mathbb{F}$ defines a representation of the Cuntz algebra $\mathcal{O}_N$.

The map $\mathbb{F}\mathbb{F}^*$

In general, we can associate a matrix with $\mathbb{F}\mathbb{F}^*$:

$$\mathbb{F}\mathbb{F}^* = (F_i F_j^*)_{i,j=1}^N,$$

and

$$F_i F_j^* = \sum_{k=1}^N (F_i F_k^*)(F_k F_j^*).$$

So, $\mathbb{F}\mathbb{F}^*$ is the identity if and only if the cross-terms for unequal indices disappear.

Connection to IFSs

Theorem

Let $(X, \mathcal{B}, \mu)$ be finite measure space, and suppose $\{\tau_1, \ldots \tau_N\}$ are measurable endomorphisms on X.
Then μ is an equal-weight equilibrium measure for the IFS generated by $\{\tau_1, \ldots \tau_N\}$ if and only if the operators $(F_1, \ldots, F_N)$ defined by

$$F_i : L^2(\mu) \to L^2(\mu)$$

$$F_i(f) = \frac{1}{\sqrt{N}} f \circ \tau_i$$

form a column isometry.

Connection to essential overlap

Theorem

Suppose $\mathbb{F} = (F_1, \ldots, F_N)$ is the column isometry defined by sufficiently nice τ maps.

Then $\mathbb{F}$ maps **onto** $\begin{pmatrix} L^2(\mu) \\ \oplus \\ \vdots \\ \oplus \\ L^2(\mu) \end{pmatrix}$ *if and only if the IFS has zero essential overlap.*

Key tools in the proof

- Partitioning $\tau(X)$ into $\{E_1, \ldots, E_k\}$ so that there are measurable mappings $\sigma_i : E_i \to X$ such that on E_i, $\sigma_i \circ \tau$ is the identity
- Calculating the Radon-Nikodym derivatives

$$\frac{d\mu \circ \tau_i^{-1}}{d\mu}$$

  - The composition operators are bounded when the RN derivatives are L^∞.
- Showing that the Radon-Nikodym derivatives are supported on the images of X under the τ maps