The intersection of the Sierpinski Carpet with straight lines.

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July 2009 / Strobl, Austria

Outline

Let F denote the Sierpinski carpet and let $E_{\theta,a} := \{(x,y) \in F : y - x\tan\theta = a\}$ denote its intersection with the line of slope θ through $(0,a)$. We shall study the dimension of $E_{\theta,a}$, $a \in [0,1]$, and $\tan\theta \in \mathbb{Q}$

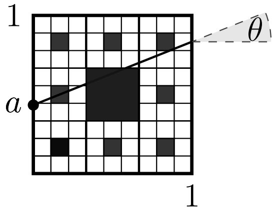

Figure: The intersection of the Sierpinski carpet with the line $y = \frac{2}{5}x + a$ for some $a \in [0,1]$.

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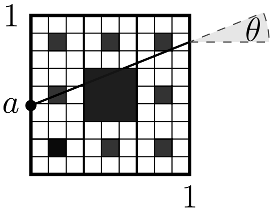

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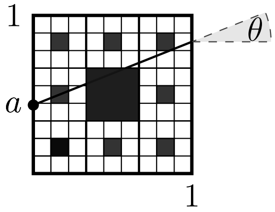

Figure: The intersection of the Sierpinski carpet with the line $y = \frac{2}{5}x + a$ for some $a \in [0,1]$.

History (old)

Theorem (Well known I.)

For all θ, for $\mathcal{L}\mathrm{eb}_1$ almost all a we have

$$\dim_{\mathrm{H}}(E_{\theta,a}) \leq \dim_{\mathrm{H}} F - 1. \tag{1}$$

Theorem (Well known II.)

$$\mathcal{L}\mathrm{eb}_2 \{(\theta, a) : \dim_{\mathrm{H}}(E_{\theta,a}) = \dim_{\mathrm{H}}(F) - 1\} > 0. \tag{2}$$

recall : F : Sierpinski carpet, $E_{\theta,a} := \{(x, y) \in F : y - x \tan\theta = a\}$

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History (recent)

Theorem (Liu, Xi and Zhao (2007))

If $\tan(\theta) \in \mathbb{Q}$ *then,*

(a) *for Lebesgue almost* a*,*
$\dim_{\mathrm{H}}(E_{\theta,a}) = \dim_{\mathrm{B}}(E_{\theta,a}) = \mathrm{const}(\theta)$.

(b) *The dimension of* $E_{\theta,a}$ *is the same for almost all* $a \in [0,1]$ *and it can be expressed as the Lyapunov exponent of a certain random matrix product divided by* $\log 3$.

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Motivation

Conjecture (Liu, Xi and Zhao (2007))

For all θ *such that* $\tan\theta \in \mathbb{Q}$*, for almost all* a *we have* $\dim_{\mathrm{H}}(E_{\theta,a}) < \dim_{\mathrm{H}} F - 1$

For $\tan\theta \in \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$, this Conjecture was verified by Liu, Xi and Zhao.

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The main theorem

We prove that the conjecture above holds:

Theorem (Manning, S. (2009))

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Thm [MS]: $\tan\theta \in \mathbb{Q} \Longrightarrow \dim_{\mathrm{H}}(E_{\theta,a}) < \dim_{\mathrm{H}} F - 1$ for a.a. a.

Some examples:

$$
\begin{aligned}
\dim_{\mathrm{H}} E_{0,1/2} &= \dim_{\mathrm{H}} E_{\pi/4,0} = \frac{\log 2}{\log 3} \\
&< \frac{\log 8}{\log 3} - 1 = \dim_{\mathrm{H}} F - 1 \\
&< \dim_{\mathrm{H}} E_{\pi/4,1/2} = \dim_{\mathrm{H}} E_{0,0} = 1.
\end{aligned}
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Thm [MS]: $\tan\theta \in \mathbb{Q} \Longrightarrow$ $\dim_{\mathrm{H}}(E_{\theta,a}) < \dim_{\mathrm{H}} F - 1$ for a.a. a.

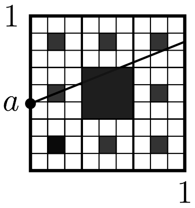
We define three matrices A_0, A_1, A_2 then we consider the Lyapunov exponent of the random matrix product

$$\gamma := \lim_{n\to\infty} \frac{1}{n} \log \|A_{i_1} \cdots A_{i_n}\|_1 ,$$

where $A_{i_k} \in \{A_0, A_1, A_2\}$ chosen independently in every step with probabilities $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Then we prove that

$$\gamma < \frac{log 8}{\log 3}.$$

$\tan\theta = 2/5$



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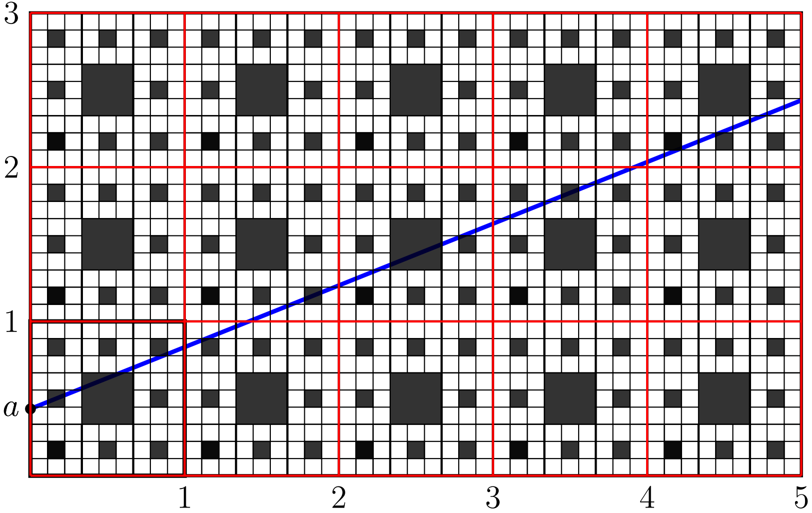

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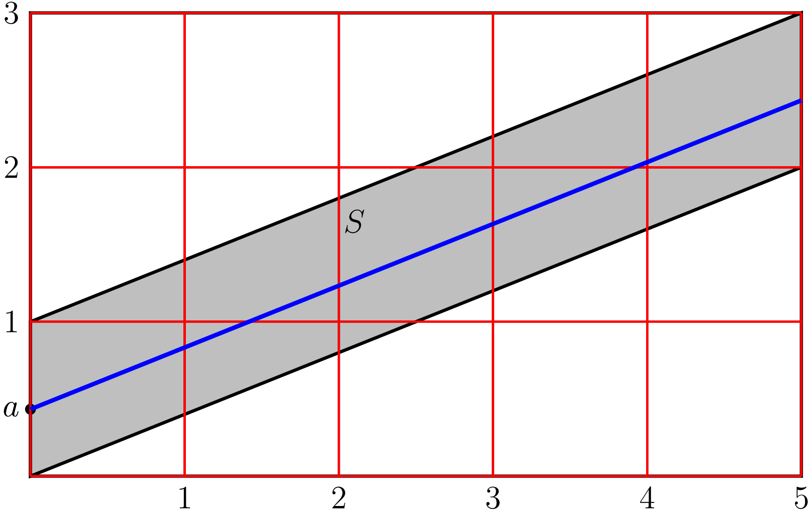

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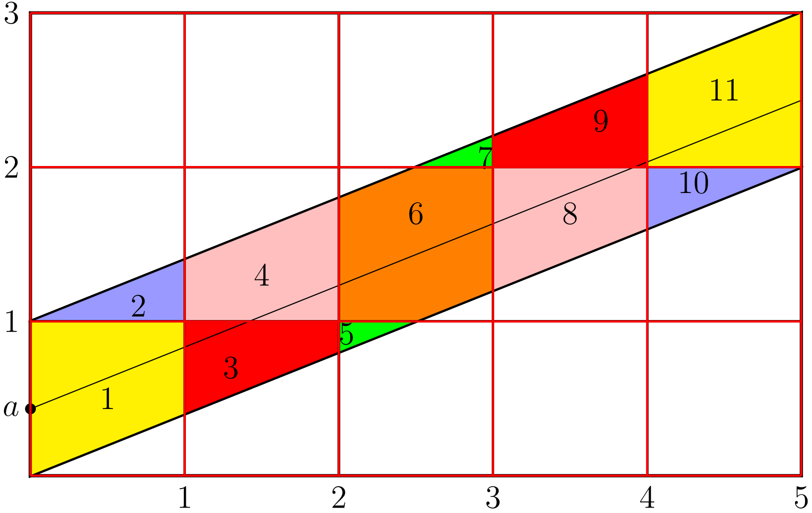

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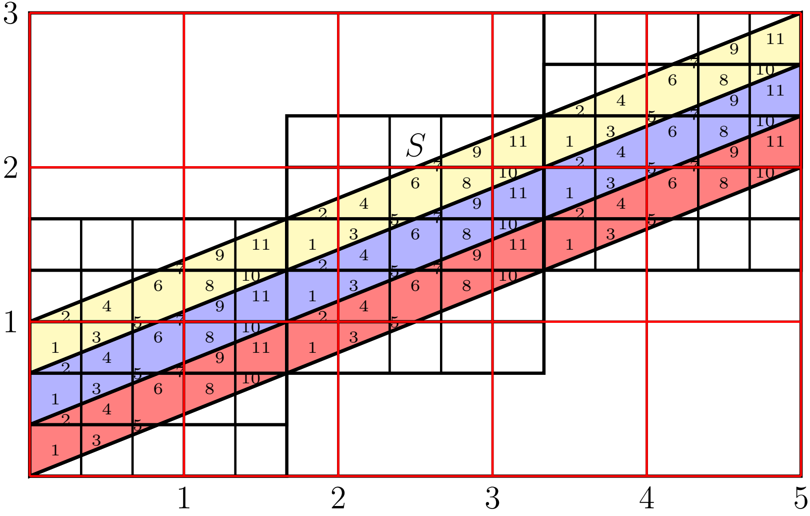

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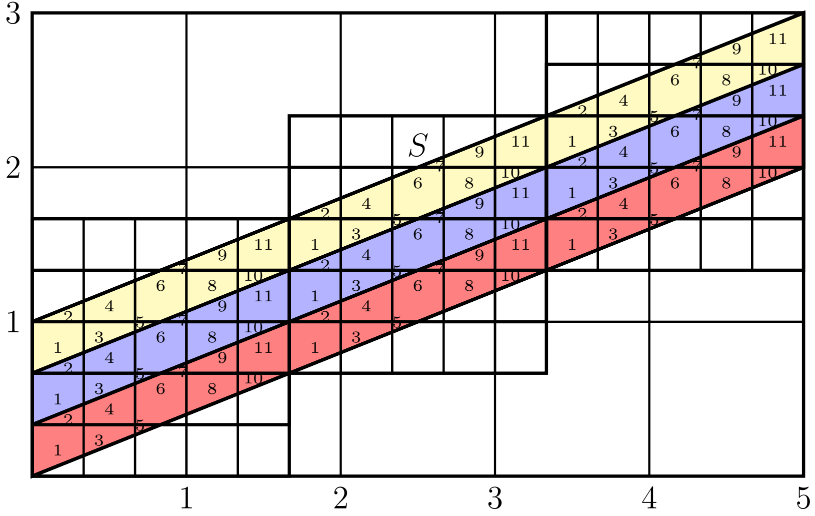

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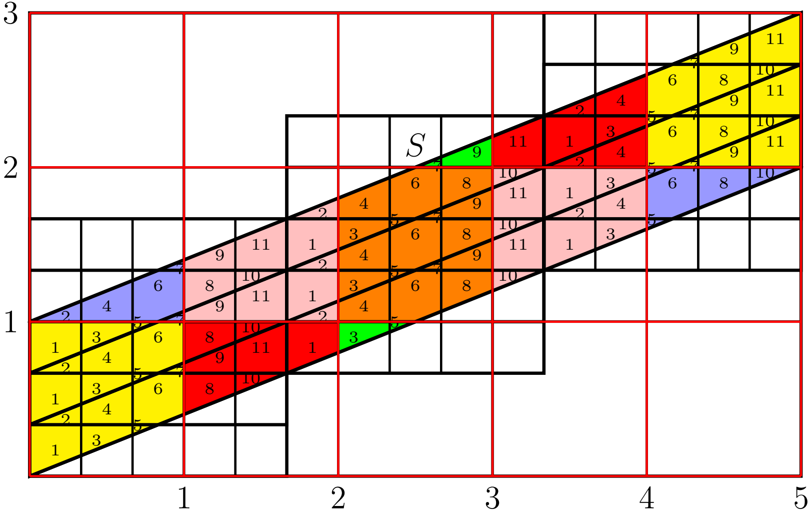

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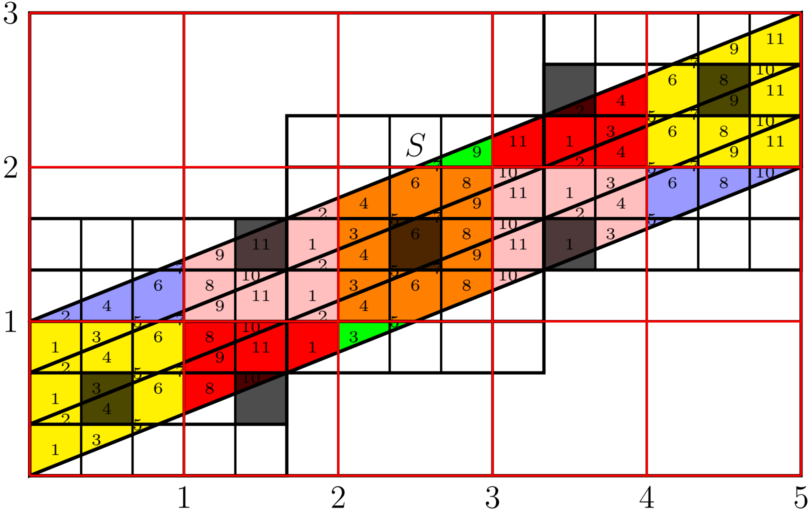

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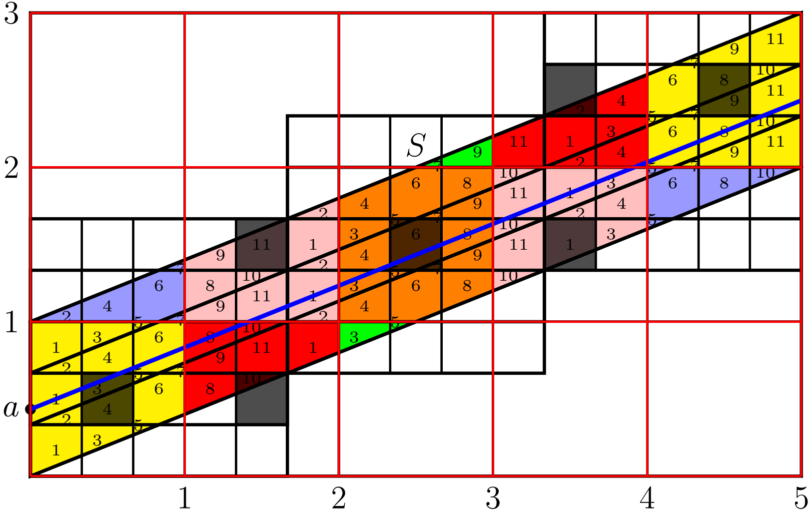
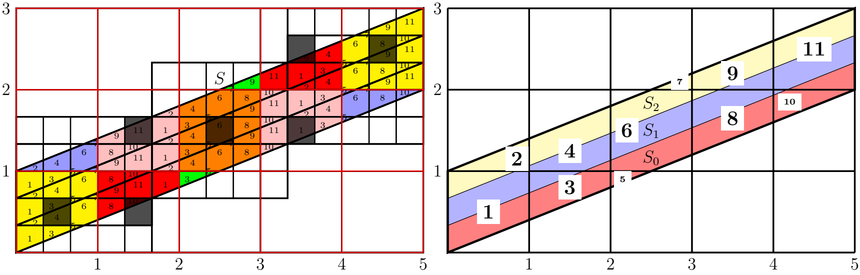


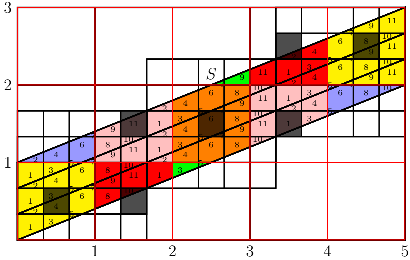
From now we always write

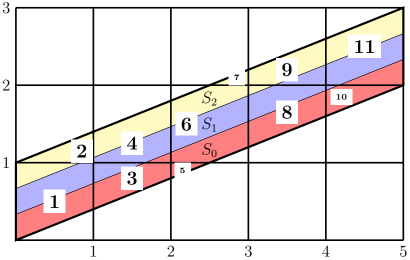
$$\frac{M}{N} := \tan\theta \qquad (M,N)=1 \qquad 3 \nmid N,$$

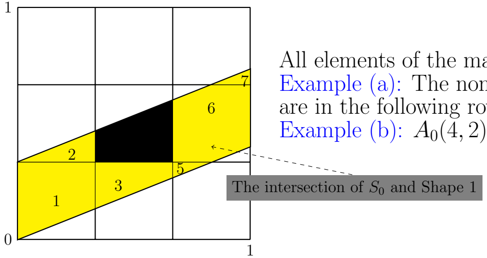
where for symmetry without loss of generality we may assume that $3 \nmid N$. (Otherwise we take N/M and change the translation parameter a appropriately.)
There are K:=2N+M-1 level zero shapes $Q_1, \ldots, Q_K$. For each "horizontal" (I mean non-vertical) stripes S_0, S_1, S_2 we define the $K \times K$ matrix A_0, A_1, A_2 respectively as follows:



$A_k(i,j) = 1$ iff the level zero shape i contains a level one shape j in stripe S







All elements of the matrices A_0, A_1, A_2 are either zero or one.
Example (a): The non zero elements of the first line of A_0 are in the following rows: $1, 2, 3, 5, 6, 7$.
Example (b): $A_0(4,2) = 1, \ \forall j \neq 2 : \ A_0(4,j) = 0$.

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$$A_0 = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ \ldots \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \ldots \end{pmatrix}.$$

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Why do we need this?

For an $a = \sum_{k=1}^{\infty} a_k \cdot 3^{-k}$, with $a_k \in \{0, 1, 2\}$:

Observation: $A_{a_1 \dots a_n}(i, j)$ is the number of level n non-deleted squares that intersect $E_{\theta,a}$. So, the number of level n-squares needed to cover $E_{\theta,a}$ is equal to $\|A_{a_1} \cdots A_{a_n}\|_1$, that is the sum of the elements of the non-negative $K \times K$ matrix $A_{a_1} \cdots A_{a_n}$. Since the size of the level n squares are $\sqrt{2} \cdot 3^{-n}$ this yields that

$$\overline{\dim}_{\mathrm{B}}(E_{\theta,a}) \leq \frac{\overbrace{\lim_{n\to\infty} \frac{1}{n} \log \|A_{a_1} \cdots A_{a_n}\|_1}^{\gamma}}{\log 3}, \qquad (3)$$

To estimate the dimension of $E_{\theta,a}$ we need to understand the exponential growth rate of the norm of $A_{a_1 \dots a_n} := A_{a_1} \cdots A_{a_n}$ which is the Lyapunov exponent of the random matrix product where each term in the matrix product is chosen from $\{A_0, A_1, A_3\}$ with probability $1/3$ independently:

$$\gamma := \lim_{n \to \infty} \frac{1}{n} \log \|A_{a_1 \dots a_n}\|_1, \text{ for a.a. } (a_1, a_2, \dots). \tag{4}$$

The limit exists (sub additive E.T.) and

$$\gamma = \lim_{n \to \infty} \frac{1}{n} \sum_{a_1 \dots a_n} \frac{1}{3^n} \log \|A_{i_1 \dots i_n}\|_1. \tag{5}$$

Essentially what we need to prove it is that

$$\gamma < \log \frac{8}{3} \tag{6}$$

holds. Namely, by (3) $\overline{\dim}_{\mathrm{B}}(E_{\theta,a}) \leq \frac{\gamma}{\log 3}$ and hence $\gamma < \log \frac{8}{3}$ is equivalent to

$$\begin{aligned} \overline{\dim}_{\mathrm{B}}(E_{\theta,a}) &\leq \frac{\gamma}{\log 3} \\ &< \frac{\log 8/3}{\log 3} = \frac{\log 8}{\log 3} - 1 = \dim_{\mathrm{H}}(F) - 1. \end{aligned}$$

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Clearly, $\gamma \leq \log \frac{8}{3}$ holds. Namely, for

$$A_s := A_0 + A_1 + A_2 :$$

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \sum_{i_1 \ldots i_n} \frac{1}{3^n} \log \|A_{i_1 \ldots i_n}\|_1$$

$$\leq \lim_{n\to\infty} \frac{1}{n} \log \frac{\sum_{i_1 \ldots i_n} \|A_{i_1 \ldots i_n}\|_1}{3^n}$$

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We needed to take higher iterates of the system (to get a system that is contracting on average in the projective distance) to prove the strict inequality.

- $\mathcal{CA}$: the set of $K \times K$ non-negative, column allowable (all columns contain non-zero elements) matrices.
- $\mathcal{CA}_p$: the set of those element of $\mathcal{CA}$ for which every row vector is either all positive or all zero.
- We prove that $\exists n_0$ and $(a'_1, \ldots, a'_{n_0}) \in \{0, 1, 2\}^{n_0}$ s.t.

$$B_1 := A_{a'_1} \cdots A_{a'_{n_0}} \in \mathcal{CA}_p.$$

Clearly, $A_{i_1} \cdots A_{i_{n_0}} \in \mathcal{CA}$ holds for all $(i_1, \ldots i_{n_0})$.

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Let $T := 3^{n_0}$, we have already defined the matrix B_1 now we define $B_2, \ldots, B_T$:

$$\{B_1, \ldots, B_T\} := \left\{A_{a_1 \ldots a_{n_0}}\right\}_{a_1 \ldots a_{n_0} \in \{0,1,2\}^{n_0}}.$$

For the vectors with all elements positive $\mathbf{x} = (x_1, \ldots, x_K) > \mathbf{0}$ and $\mathbf{y} = (y_1, \ldots, y_K) > \mathbf{0}$ we define the pseudo-metric

$$d(\mathbf{x}, \mathbf{y}) := \log \left[\frac{\max_i (x_i/y_i)}{\min_j (x_j/y_j)}\right].$$

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$$\Delta := \left\{ \mathbf{x} = (x_1, \ldots, x_K) \in \mathbb{R}^K : x_i > 0 \text{ and } \sum_{i=1}^{K} x_i = 1 \right\}$$

We call it projective distance. For all $A \in \mathcal{CA}$ we define

$$\widetilde{A} : \Delta \to \Delta \qquad \widetilde{A}(\mathbf{x}) := \frac{\mathbf{x}^T \cdot A}{\|\mathbf{x}^T \cdot A\|_1}.$$

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$$\tau_B(A) := \sup_{\mathbf{x},\mathbf{y} \in \Delta,\ \mathbf{x} \neq \mathbf{y}} \frac{d(\mathbf{x}^T \cdot A, \mathbf{y}^T \cdot A)}{d(\mathbf{x}, \mathbf{y})}.$$

Lemma (Well known)

(a) *For* $\forall i = 1, \ldots, 3^n$: $\tau(B_i) \leq 1$.

(b) *The map* B_1 *is a strict contraction in the projective distance:*

$$h := \tau(B_1) < 1.$$

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Corollary of the Lemma:

So, the following IFS acting on the non-compact metric space (Δ, d) is contracting on average:

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in the strong sense that the average of the Lipschitz constants is less than one.

recall : Δ : is the simplex:

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Definition

Suggested by a paper of Kravchenko (2006), on the complete metric space (Δ, d) we write $M(\Delta)$ for the set of all probability measures on Δ for which $\mu(\phi) < \infty$ holds for all real valued Lipschitz functions ϕ defined on (Δ, d). After Kantorovich, Rubinstein we define the distance of $\mu, \nu \in M(\Delta)$ by

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We introduce the operator $\mathcal{F} : M(\Delta) \to M(\Delta)$

$$\mathcal{F}\nu(H) := \frac{1}{T} \cdot \sum_{i=1}^{T} \nu\left(\widetilde{B}_i^{-1}(H)\right).$$

for a Borel set $H \subset \Delta$. Using $\nu \in M(\Delta)$, for every Lipschitz function ϕ we have

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(a) *$\mathcal{F}$ is a contraction on the metric space $(M(\Delta), L)$.*

(b) *There is a unique fixed point $\nu \in M(\Delta)$ of $\mathcal{F}$ and for all $\mu \in M(\Delta)$ we have $L(\nu, \mathcal{F}^n\mu) \to 0$.*

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$$\nu(\phi) = \frac{1}{T^n} \cdot \sum_{i_1 \dots i_n} \nu(\phi \circ \widetilde{B}_{i_1 \dots i_n}). \tag{7}$$

holds for all Lipschitz functions ϕ and $n \geq 1$.

Following an idea of Furstenberg, it is a key point of our argument that we would like to give an integral representation of the Lyapunov exponent γ_B as an integral of a function φ to be introduced below against the measure ν.

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Lemma

Let γ_B be the Lyapunov exponent of the random matrix product formed from the matrices $B_1, \ldots, B_T$ taking each of the matrices with equal weight independently in every step. Then

$$n_0\gamma = \gamma_B = \int_\Delta \varphi(\mathbf{x}) d\nu(\mathbf{x})$$

where $\varphi : \Delta \to \mathbb{R}$ is defined by

$$\varphi(\mathbf{x}) := \frac{1}{T} \cdot \sum_{k=1}^{T} \log \|\mathbf{x} \cdot B_k\|_1, \qquad \mathbf{x} \in \Delta. \tag{8}$$

recall: ν is the unique invariant measure for the IFS $\left\{\widetilde{B}_1, \ldots, \widetilde{B}_m\right\}$

A good piece of news:

Lemma

We have $\mathrm{Lip}(\varphi) \leq 1$ *on the metric space* (Δ, d)*.*

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We need to prove that:

$$\gamma_B < n_0 \cdot \log \frac{8}{3} \tag{9}$$

where $\gamma_B = n_0 \cdot \gamma$ is the Lyapunov exponent for the random matrix product formed from the matrices $B_1, \ldots, B_T$ each chosen independently with equal probabilities.

Let $\mathbf{w} \in \mathbb{R}^K$ be the center of the simplex Δ:

$$\mathbf{w} := \frac{1}{K} \cdot \mathbf{e} \text{ where } \mathbf{e} := (1, \ldots, 1) \in \mathbb{R}^K.$$

We define the sequence of measures $\nu_n \in \mathcal{M}^1$ by $\nu_0 := \delta_{\mathbf{w}}$ and for $H \subset \Delta$:

$$\nu_n(H) := (\mathcal{F}^n \nu_0)(H) = \frac{1}{T^n} \cdot \sum_{i_1 \ldots i_n} \nu_0(\widetilde{B}^{-1}_{i_1 \ldots i_n}(H)),$$

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We prove that $\exists \varepsilon'$ s.t. for every m big enough:

$$\int_\Delta \varphi(\mathbf{x}) d\nu_m(\mathbf{x}) = \frac{1}{T^m} \cdot \sum_{|\mathbf{i}|=m} \frac{1}{T} \sum_{j=1}^{T} \log \frac{\|B_j \cdot B_{\mathbf{i}}\|_1}{\|B_{\mathbf{i}}\|_1}$$

$$\leq n_0 \cdot \log \frac{8}{3} - \varepsilon'$$

Then

$$\lim_{n\to\infty} \int_\Delta \varphi(\mathbf{x}) d\nu_n(\mathbf{x}) = \int_\Delta \varphi(\mathbf{x}) d\nu(\mathbf{x}) = \gamma_B$$

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On the complete separable metric space (X, ρ) let $\mathcal{M}^1$ be the set of Borel regular probability measures on X with bounded support and let

$$L(\mu, \nu) := \sup \{\mu(\phi) - \nu(\phi) | \phi : \Delta \to \mathbb{R},\ \mathrm{Lip}(\phi) \leq 1\} .$$

In the paper Hutchinson 1981 Indiana Math. J. on p.733 it is claimed that for a strictly contracting IFS (all Lipschitz constant are smaller than one) for given weights there is a unique invariant measure. Although it is not spelled out directly, but the proof makes use of the claim that the metric space $(\mathcal{M}^1, L)$ is complete. However, this is false since whenever (X, ρ) is unbonded then $(\mathcal{M}^1, L)$ is not complete. (Counter example with discrete measures.)

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Kravchenko remarks the following:
The weak convergence of the measures is related to the convergence of the integrals of bounded continuous functions w.r.t. these measures. If we restrict the weak convergence of the measures to $M(X)$ then we get the same topology as given by the metric L if X is bounded. If X is unbounded then the topology given by L is strictly finer than the weak topology restricted to $M(X)$.