The k-bonacci Mandelbrot set

Víctor Sirvent and Jörg Thuswaldner

Introduction

The Mandelbrot set for a pair of linear maps

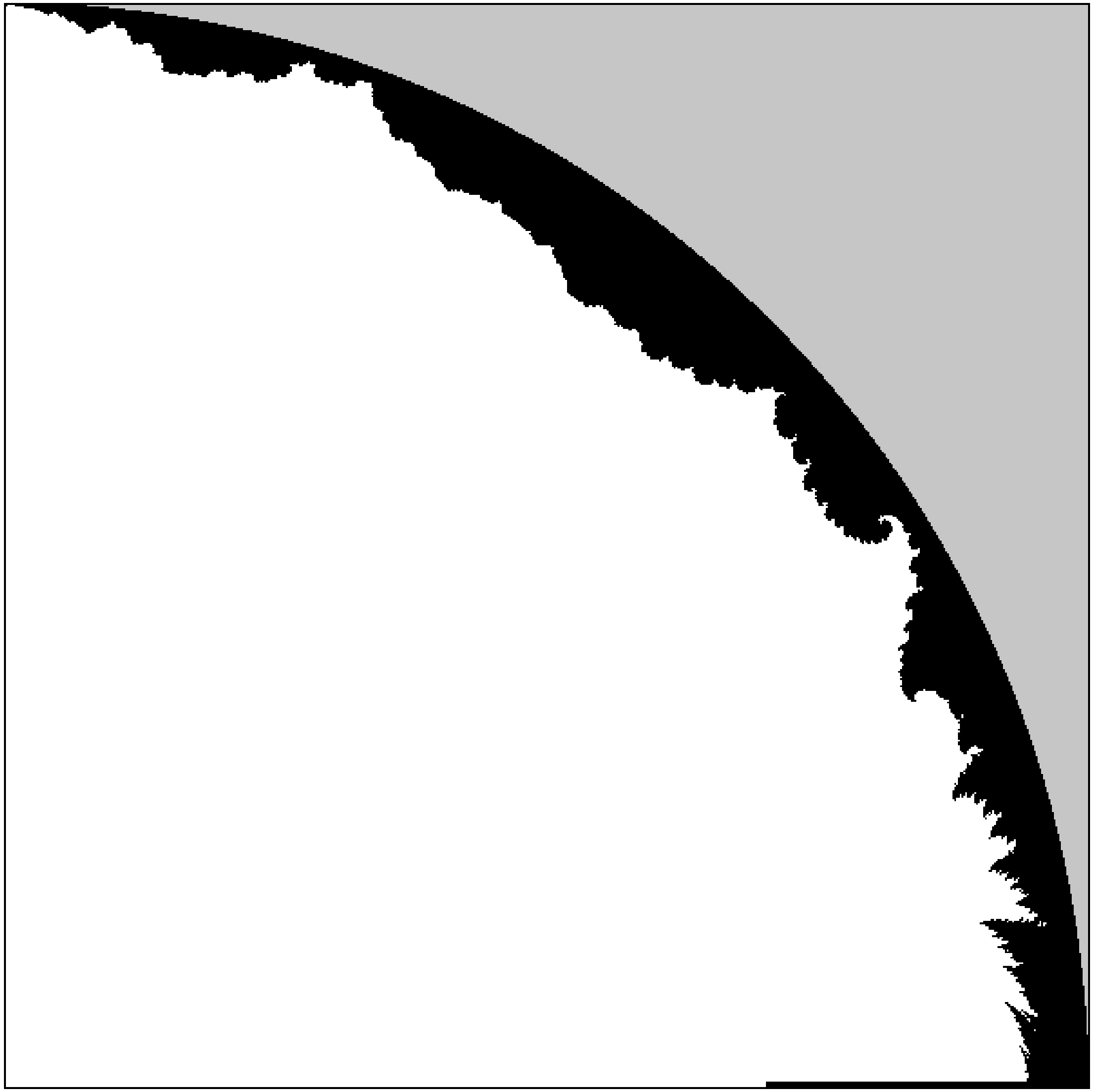
(Barnsley)

$$h_0(z) = \beta z, \quad h_1(z) = \beta z + 1$$

This IFS has an attractor A_β.
The Mandelbrot set for a pair of linear maps is

$$\{\beta \in \mathbb{D} \,:\, A_\beta \text{ is connected } \}$$

The properties of this set has been studied by: Barnsley, Harrington, Bousch, Bandt, Solomyak, Xu, Shmerkin.



Here we consider the IFS:

$$f_0(z) = \beta z, \quad f_1(z) = \beta^2 z + 1.$$

Its attractor is

$$\mathfrak{R}(\beta) = \left\{ \sum_{i=0}^{\infty} a_i \beta^i \ : a_0 \dots a_n \in \mathcal{L} \text{ holds for each } n \right\},$$

where $\mathcal{L}$ is the *Fibonacci or golden-mean language*.

If $\beta = \tau = \frac{1-\sqrt{5}}{2}$ then $\mathfrak{R}(\tau) = [-1, -\frac{1}{\tau}]$.

Fibo-Mandelbrot Set

$$\mathcal{M} = \mathcal{M}(2) := \{\beta \in \mathbb{D} \ : \ \mathfrak{R}(\beta) \text{ is connected}\}$$

Topological properties of $\mathcal{M}(2)$?

Fibo-Mandelbrot Set

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Topological properties of $\mathcal{M}(2)$?

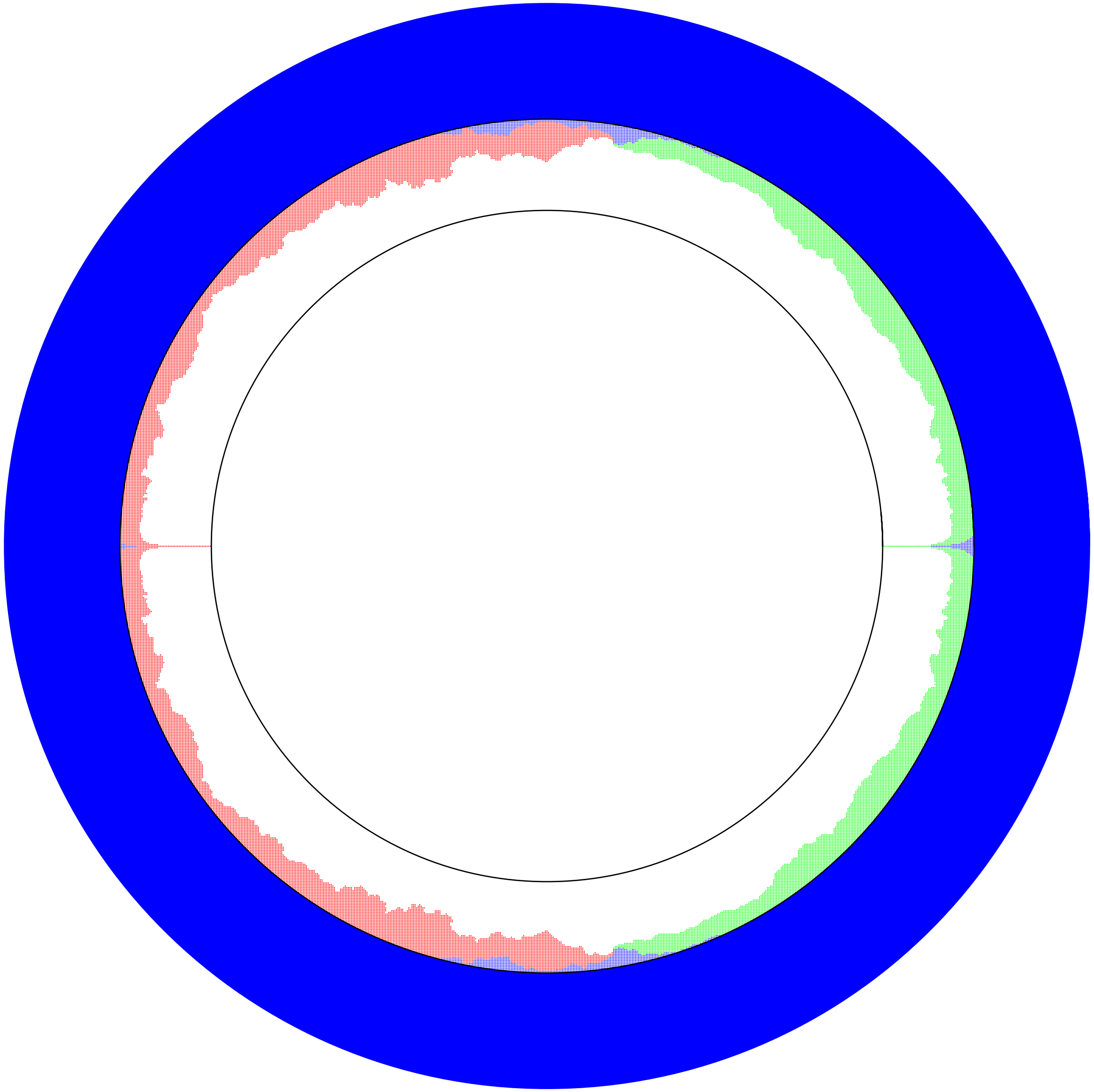
Generalization to the k-bonacci IFS:

$$f_0(z) = \beta z,\ f_1(z) = 1+\beta^2 z,\ \ldots,\ f_{k-1}(z) = 1+\beta+\cdots+\beta^{k-2}+\beta^k z.$$

$$\mathfrak{R}(\beta) = \left\{ \sum_{i=0}^{\infty} a_i \beta^i \ : a_0 \ldots a_n \in \mathcal{L}(k) \text{ holds for each } n \right\},$$

here $\mathcal{L}(k)$ is the k-bonacci language and we define $\mathcal{M}(k)$ in a similar manner.

Fibo-Mandelbrot Set



Properties of $\mathcal{M}(2)$

- If $\beta < |\tau|$ then $\mathfrak{R}(\beta)$ is totally disconnected.
- If $\beta > \sqrt{|\tau|}$ then $\mathfrak{R}(\beta)$ is connected.
- In other words

$$\{\beta \in \mathbb{D} : |\beta| > \sqrt{|\tau|}\} \subset \mathcal{M} \subset \{\beta \in \mathbb{D} : |\beta| > |\tau|\}.$$

Characterization of $\mathcal{M}$

$$\mathcal{M} = \left\{ \beta \in \mathbb{D} \ : \exists (c_i)_{i \geq 2} \in \widetilde{\mathcal{L}}^{\infty} \text{ and } a \in \{0,1\} \text{ such that } 1 - a\beta + \sum_{i=2}^{\infty} c_i \beta^i = 0 \right\}$$

where $\widetilde{\mathcal{L}}$ is the difference language of $\mathcal{L}$.

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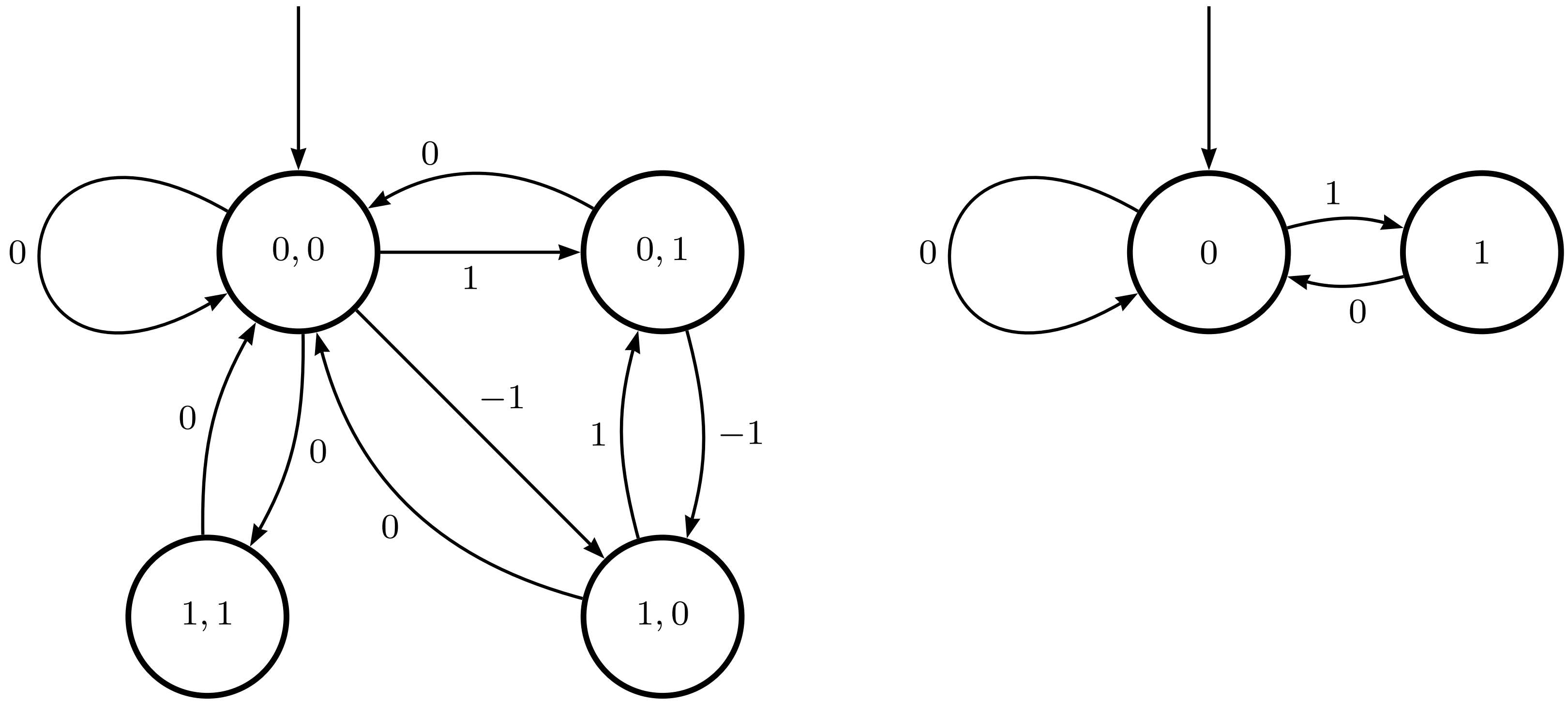
The Fibonacci automaton and its difference automaton



Theorem 1. *The set $\mathcal{M}(k)$ is connected and locally connected.*

We adapt Bousch's proof to our case.

Open problems

- Generalize previous constructions and results for sofic systems, *i.e.*, when $\mathcal{L}$ is given by a sofic system.
- Does $\mathcal{M}(k)$ have “lakes”?
- Is $\mathcal{M}(k)$ (apart from the obvious antennas) the closure of its interior? Can one describe the “root” of the antennas as it was done for the case of the full shift by Solomyak?
- If $\beta \in \mathrm{int}(\mathcal{M})$ then the IFS $\{f_0, f_1\}$ has overlaps?
- Describe the set of all parameters β for which $\mathfrak{R}_\beta$ is not totally disconnected.

Open problems — cont.

- Do there exist cusps?
- Characterize the Julia sets that are “dendrites”?
- Characterize the Julia sets having nonempty interior?
- Characterize the Julia sets that are homeomorphic to a closed disk?