Analysis of Fractals, Image Compression and Entropy Encoding

Myung-Sin Song

Southern Illinois University Edwardsville

Jul 10, 2009

Joint work with Palle Jorgensen.

Outline

1. Signal and Image processing,
2. Selfsimilarity, Computational Features
3. Operator Theoretic Models
4. Slanted Matrix Representations
5. Image Decomposition using Forward Wavelet Transform
6. Entropy Encoding and Karhunen-Loève Transform
7. Wavelets and Fractals, and Fractal image processing

1. Signal and Image processing

(a) A systematic study of bases in Hilbert spaces built on fractals suggests a common theme: A hierarchical multiscale structure. A well-known instance of the self-similarity is reflected in the scaling rules from wavelet theory.

(b) The best known instances: the dyadic wavelets in $L^2(\mathbb{R})$, built by two functions φ and ψ; subject to the relation

$$\varphi(x) = 2\sum h_n \varphi(2x - n), \text{ and } \psi(x) = 2\sum g_n \varphi(2x - n). \tag{1}$$

where (h_n) and (g_n) are fixed and carefully chosen sequences.

1. Signal and Image processing - cont'd

(d) The function φ is called the scaling function, or the father function, and ψ is called the mother function.

(e) The best known choice of pairs of filter coefficients (h_n), (g_n) is the following: Pick $(h_n) \subset \mathbb{R}$ subject to the two conditions $\sum_{n \in \mathbb{Z}} h_n = 1$ and $\sum_{n \in \mathbb{Z}} h_n h_{n+2l} = \frac{1}{2}\delta_{0,l}$. Then set $g_n := (-1)^n h_{1-n}$, $n \in \mathbb{Z}$.

(f) The convention is that (h_n) is 0 outside some specified range.

1. Signal and Image processing - cont'd

(g) The associated double indexed family $\psi_{jk}(x) = 2^{i/2}\psi(2^j x - k), j, k, \in \mathbb{Z}$ will be a wavelet basis for $L^2(\mathbb{R})$.

(h) This is the best known wavelet construction also known by the name multi-resolution analysis (MRA). The reason for this is that the father function φ generates a subspace V_0 of $L^2(\mathbb{R})$ which represents a choice of resolution for wavelet decomposition.

1. Signal and Image processing - cont'd

Definition

Take V_0 to be the closed span of all the translates $(\varphi(\cdot - k))$, $k \in \mathbb{Z}$ in $L^2(\mathbb{R})$. From (1), it follows that the scaling operator $Uf(x) = 2^{-1/2} f(\frac{x}{2})$ maps the space V_0 into itself; and that $U\psi \in V_0$.

With suitable modification this idea also works for wavelet bases in $L^2(\mathbb{R}^d)$, and in Hilbert spaces built on fractals.

2.1 Selfsimilarity

(a) For Julia sets X in complex analysis for example, U could be implemented by a rational function $z \mapsto r(z)$.

(b) When r is given, X will be a compact subset of $\mathbb{C}$ which is determined by the dynamics of $r^n = \underbrace{r \circ \cdots \circ r}_{n \text{ times}}$. Specifically,

$$\mathbb{C} \backslash X = \cup \{ \mathcal{O} | \mathcal{O} \text{ open}, (r^{(n)} | \mathcal{O}) \text{ is normal} \}. \tag{2}$$

(c) Interested in showing that these non-linear fractals are related to more traditional wavelets, i.e., those of $L^2(\mathbb{R}^d)$. We want to extend the $\mathbb{R}^d$-analysis both to fractals and to discrete hierarchical models.

2.2 Computational Features

(a) Approximation of the father or mother functions by subdivision schemes.

(b) Matrix formulas for the wavelet coefficients. For fractals, L^2-convergence is more restrictive than is the case for $L^2(\mathbb{R}^d)$-wavelets.

A unifying approach to wavelets, dynamical systems, iterated function systems, self-similarity and fractals may be based on the systematic use of operator analysis and representation theory.

3.1 Operator Theoretic Models

(a) Motivation: hierarchical models and multiscaling, operators of multiplication, and dilations, and more general weighted composition operators are studied. Scaling is implemented by non-linear and non-invertible transformations. This generalizes affine trasformations of variables from wavelet analysis and analysis on affine fractals.

(b) The properties of dynamical and iterated function systems, defined by these transformations, govern the spectral properties and corresponding subspace decompositions.

3.1 Operator Theoretic Models - cont'd

(c) The interplay between dynamical and iterated function systems and actions of groups and semigroups on one side, and operator algebras on the other side, yield new results and methods for wavelets and fractal analysis and geometry.

(d) Wavelets, signals and information may be realized as vectors in a real or complex Hilbert space. In the case of images, this may be worked out using wavelet and filter functions, e.g. corresponding to ordinary Cantor fractal subsets of $\mathbb{R}$, as well as for fractal measure spaces of Sierpinski Gasket fractals.

3.2 Operators and Hilbert Space

(a) Operator algebra constructions of covariant representations are used in the analysis of orthogonality in wavelet theory, in the construction of super-wavelets, and orthogonal Fourier bases for affine fractal measures.

(b) In signal processing, time-series, or matrices of pixel numbers may similarly be realized by vectors in Hilbert space $\mathcal{H}$.

(c) In signal/image processing, because of aliasing, it is practical to generalize the notion of ONB, and this takes the form of what is "a system of frame vectors."

3.2 Operators and Hilbert Space - cont'd

(d) One particular such ONB goes under the name "the Karhunen-Loève basis."

(e) Motivation comes from the consideration of the optimal choices of bases for certain analogue-to-digital (A-to-D) problems we encountered in the use of wavelet bases in image-processing.

3.2 Operators and Hilbert Space - cont'd

Definition

For every finite n, a representation of the Cuntz algebra $\mathcal{O}_n$ is a system of isometries $S_i : \mathcal{H} \to \mathcal{H}$ such that

(a) $S_i^* S_j = \delta_{ij} I$; orthogonality, and

(b) $\sum_i S_i S_i^* = I$ (perfect reconstruction).

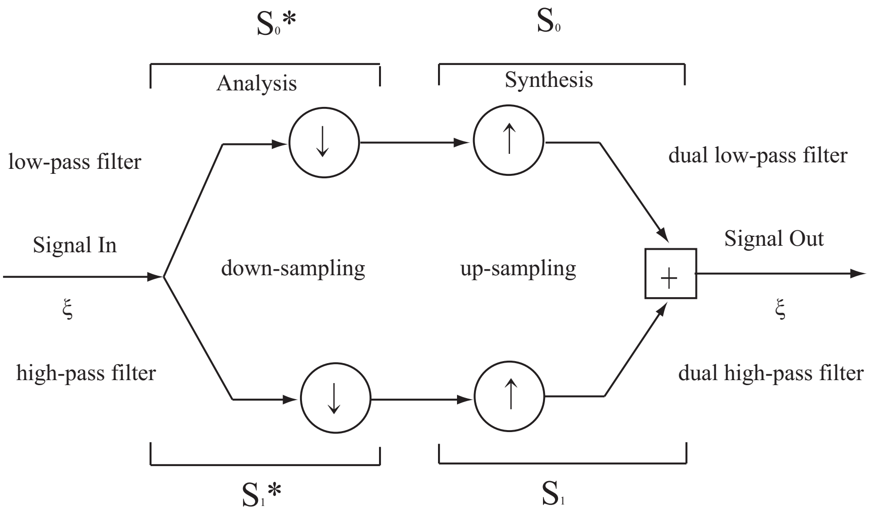


Figure: Examples of operators in signal image processing.

4. Slanted Matrix Representations

Definition

If $(h_n)_{n\in\mathbb{Z}}$ is a double infinite sequence of complex numbers, i.e., $h_n \in \mathbb{C}$, for all $n \in \mathbb{Z}$; set

$$(S_0 x)(m) = \sqrt{2} \sum_{n\in\mathbb{Z}} h_{m-2n} x(n) \tag{3}$$

and adjoint

$$(S_0^* x)(m) = \sqrt{2} \sum_{n\in\mathbb{Z}} \bar{h}_{n-2m} x(n); \text{ for all } m \in \mathbb{Z}. \tag{4}$$

4. Slanted Matrix Representations - cont'd

Then

(a) The $\infty \times \infty$ matrix representations (3) and (4) have the following slanted forms

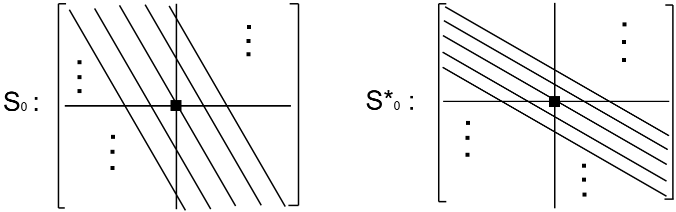


Figure: S_0 and S_0^*.

4. Slanted Matrix Representations - cont'd

(b) The set of non-zero numbers in $(h_n)_{n\in\mathbb{Z}}$ is finite if and only if the two matrices in Figure are *banded*.

(c) Relative to the inner product

$$\langle x|y\rangle_{l^2} := \sum_{n\in\mathbb{Z}} \bar{x}_n y_n \text{ in } l^2$$

(i.e., conjugate-linear in the first variable), the operator S_0 is *isometric* if and only if

$$\sum_{n\in\mathbb{Z}} \bar{h}_n h_{n+2p} = \frac{1}{2}\delta_{0,p}, \text{ for all } p \in \mathbb{Z}. \quad (5)$$

4. Slanted Matrix Representations - cont'd

(d) If (5) holds, and

$$(S_1x)(m) = \sqrt{2}\sum_{n\in\mathbb{Z}} g_{m-2n}x(n), \tag{6}$$

then

$$S_0S_0^* + S_1S_1^* = I_{l^2} \tag{7}$$

$$S_k^*S_l = \delta_{k,l}I_{l^2} \text{ for all } k,l \in \{0,1\} \tag{8}$$

(the Cuntz relations) holds for

$$g_n := (-1)^n\bar{h}_{1-n},\ n \in \mathbb{Z}.$$

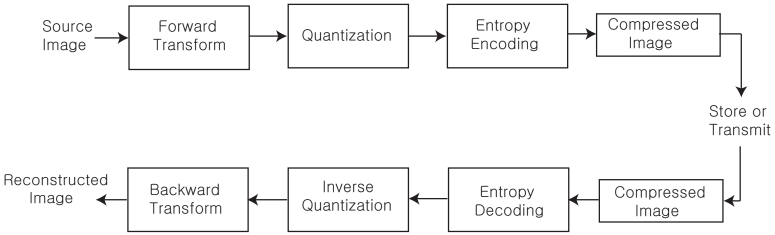


Figure: Outline of the wavelet image compression process.

5. Image Decomposition using Forward Wavelet Transform

A 1-level wavelet transform of an $N \times M$ image can be represented as

$$\mathbf{f} \mapsto \begin{pmatrix} \mathbf{a}^1 & | & \mathbf{h}^1 \\ -- & & -- \\ \mathbf{v}^1 & | & \mathbf{d}^1 \end{pmatrix}$$

where the subimages $\mathbf{h}^1, \mathbf{d}^1, \mathbf{a}^1$ and $\mathbf{v}^1$ each have the dimension of $N/2$ by $M/2$.

5. Image Decomposition using Forward Wavelet Transform-cont'd

$$\mathbf{a}^1 = V_m^1 \otimes V_n^1 : \varphi^A(x,y) = \varphi(x)\varphi(y)$$
$$= \sum_i \sum_j h_i h_j \varphi(2x-i)\varphi(2y-j)$$
$$\mathbf{h}^1 = W_m^1 \otimes V_n^1 : \psi^H(x,y) = \psi(x)\varphi(y)$$
$$= \sum_i \sum_j g_i h_j \varphi(2x-i)\varphi(2y-j)$$
$$\mathbf{v}^1 = V_m^1 \otimes W_n^1 : \psi^V(x,y) = \varphi(x)\psi(y)$$
$$= \sum_i \sum_j h_i g_j \varphi(2x-i)\varphi(2y-j)$$
$$\mathbf{d}^1 = W_m^1 \otimes W_n^1 : \psi^D(x,y) = \psi(x)\psi(y)$$
$$= \sum_i \sum_j g_i g_j \varphi(2x-i)\varphi(2y-j)$$

φ : the father function in sense of wavelet.

ψ : is the mother function in sense of wavelet.

V space : the average space and the from multiresolution analysis (MRA).

W space : the difference space from MRA.

h : low-pass filter coefficients

g : high-pass filter coefficients.

Test Image

Figure: Prof. Jorgensen in his office.

First-level Decomposition

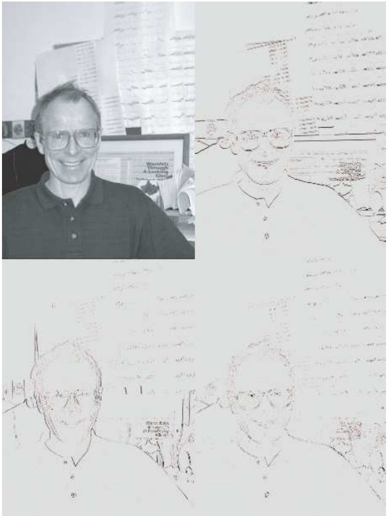


Figure: 1-level Haar Wavelet Decomposition of Prof. Jorgensen

Second-level Decomposition

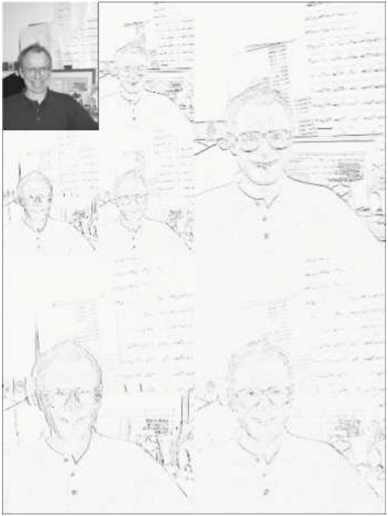


Figure: 2-level Haar Wavelet Decomposition of Prof. Jorgensen

6.1 Entropy Encoding

- Entropy encoding further compresses the quantized values in lossless manner which gives better compression in overall.
- It uses a model to accurately determine the probabilities for each quantized value and produces an appropriate code based on these probabilities so that the resultant output code stream will be smaller than the input stream.

6.1 Entropy Encoding-Example

An example with letters in the text would better depict how the mechanism works. Suppose we have a text with letters a, e, f, q, r with the following probability distribution:

Letter	Probability
a	0.3
e	0.2
f	0.2
q	0.2
r	0.1

Example-cont'd

Then applying the Shannon-Fano entropy encoding scheme on the above table gives us the following assignment.

Letter	Probability	code
a	0.3	00
e	0.2	01
f	0.2	100
q	0.2	101
r	0.1	110

Note that instead of using 8-bits to represent a letter, 2 or 3-bits are being used to represent the letters in this case.

6.2 Karhunen-Loève Transform

- Karhunen-Loève transform is an operator theoretic tool which has proved effective and versatile in the analysis of stochastic processes (X_t).
- The starting point in this is a spectral analysis of the correlations $E(X_t X_s)$. In models, this may represent, for example, correlations of pixel values.
- The K-L analysis involves a variety of choices of bases $(\psi_l(t))$, including wavelet bases, and it leads to a sequence (Z_n) of independent random variables, and an associated K-L expansion of the initial process (X_t).

6.2 Karhunen-Loève Transform - cont'd

- Given the values of their neighbors, pixels in smooth regions can be predicted with substantial accuracy, so the independent storage of pixels is unnecessary.
- Exploiting this spatial redundancy (correlation between neighboring pixel values) enables us to acquire a considerable improvement in performance over entropy coding alone.
- Applying K-L transform to an image yields a set of transform coefficients which are de-correlated, i.e., the first-order linear redundancy in the pixels are eliminated.

6.2 Description of the Algorithm for Karhunen-Loève transform entropy encoding

1. Perform the wavelet transform for the whole image. (i.e., wavelet decomposition.)
2. Do quantization to all coefficients in the image matrix, except the average detail.
3. Subtract the mean: Subtract the mean from each of the data dimensions. This produces a data set whose mean is zero.
4. Compute the covariance matrix

$$cov(X, Y) = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{n}.$$

6.2 Description of the Algorithm for Karhunen-Loève transform entropy encoding - cont'd

5. Compute the eigenvectors and eigenvalues of the covariance matrix.
6. Choose components and form a feature vector(matrix of vectors),

$$(eig_1, ..., eig_n).$$

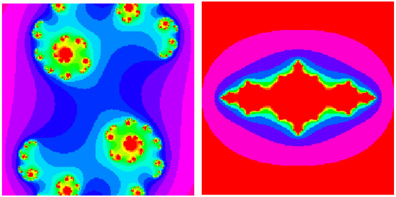
   Eigenvectors are listed in decreasing order of the magnitude of their eigenvalues. Eigenvalues found in step 5 are different in values. The eigenvector with highest eigenvalue is the principle component of the data set.
7. Derive the new data set.

$$\text{Final Data} = \text{Row Feature Matrix} \times \text{Row Data Adjust}.$$

6.2 Description of the Algorithm for Karhunen-Loève transform entropy encoding - cont'd

- Row Feature Matrix : the matrix that has the eigenvectors in its rows with the most significant eigenvector (i.e., with the greatest eigenvalue) at the top row of the matrix.
- Row Data Adjust : the matrix with mean-adjusted data transposed. That is, the matrix contains the data items in each column with each row having a separate dimension.

7.1 Wavelets and Fractals



(a) The simplest Julia sets come from a one parameter family of quadratic polynomials $\varphi_c(z) = z^2 + c$, where z is a complex variable and where c is a fixed parameter.

(b) Consider the two branches of the inverse $\beta_{\pm} = z \mapsto \pm\sqrt{z - c}$. Then J_c is the unique minimal non-empty compact subset of $\mathbb{C}$, which is invariant under $\{\beta_{\pm}\}$.

(c) Interested in adapting and modifying the Haar wavelet, and the other wavelet algorithms to the Julia sets.

7.1 Wavelets and Fractals - cont'd

(d) There was an initition of wavelet transforms for complex fractals.

(e) These transforms have some parallels to traditional affine fractals, but subtle non-linearities precluded from writing down an analogue of Haar wavelets in these different settings.

(f) Want to develop more refined algorithms taking these difficulties into account.

7.1 Wavelets and Fractals - cont'd

A successful harmonic analysis on Julia sets, with their Brolin measures, is likely to be more difficult than is the corresponding situation for the affine fractals with their Hutchinson measures (such as Cantor and Sierpinski constructs).
This difficulty in an analysis of Julia sets appears both in wavelet constructions and in our search for Fourier bases.
The reason is that Julia iterations are by non-linear mappings, while affine fractals are amenable to linear tools, or rather systems of affine maps.
Still there is hope for the more non-linear case because of intrinsic selfsimilarity.

7.2 Fractal Image Processing

(a) Unlike wavelets, fractal coders store images as a fixed points of maps on the plane instead of a set of quantized transform coefficients.

(b) Fractal compression is related to vector quantization, but fractal coders use a self-referential vector codebook, drawn from the image itself, instead of a fixed codebook.

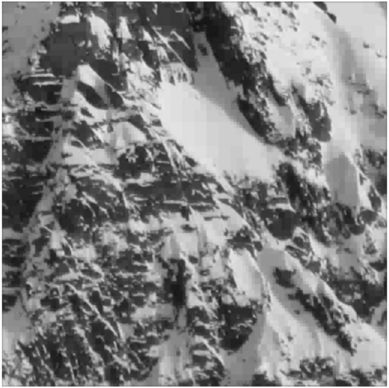
(c) IFS theory motivates a broad class fractal compression schemes but it does not show why particular fractal schemes work well.

(d) A wavelet-based framework for analyzing fractal block coders would simplify the analysis of these codes considerably and give a clear picture of why they are effective.

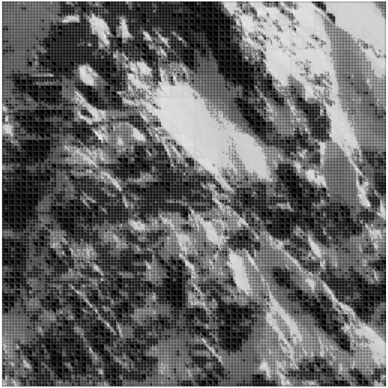
7.3 Image Decomposition using real Sierpinski-Gasket filter

By Jonas D'Andrea, Kathy Merrill and Judy Packer **(next 4 slides)**

reconstruction using 10% of transform coefficients

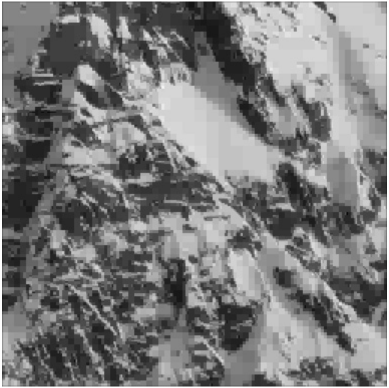


Haar filter

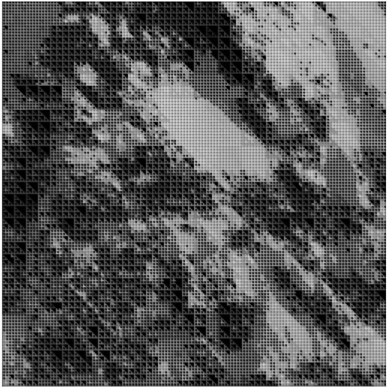


real SG filter

reconstruction using 3% of transform coefficients

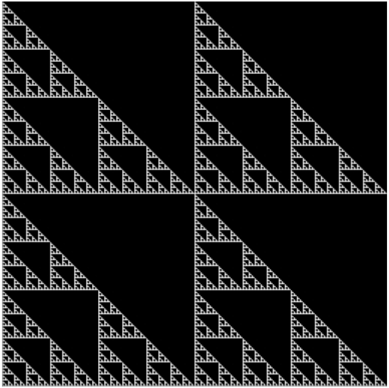


Haar filter

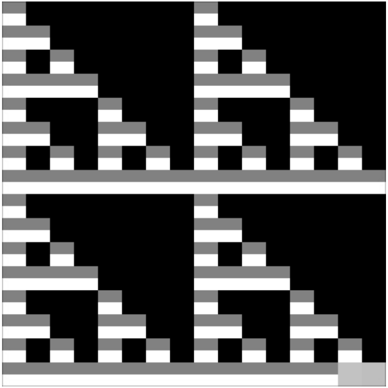


real SG filter

reconstruction using 0.1% of transform coefficients

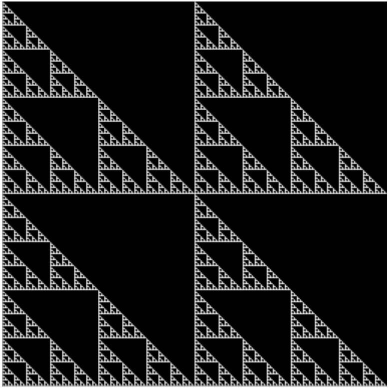


real SG filter
same as original

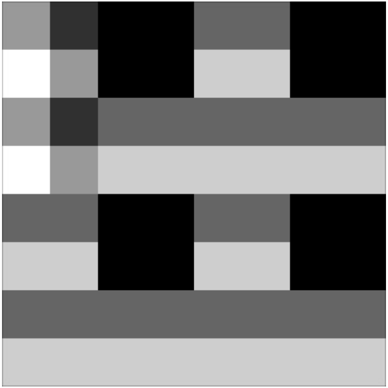


Haar filter

reconstruction using 0.01% of transform coefficients



real SG filter
same as original



Haar filter

References

- P. E. T. Jorgensen and M.-S. Song, “Analysis of Fractals, Image Compression, Entropy Encoding, Karhunen-Loève Transforms” Acta Applicandae Mathematica, Springer, 2009.
- P. E. T. Jorgensen and M.-S. Song, “Entropy Encoding, Hilbert Space and Karhunen-Loève Transforms” J. Math. Phys. **48** 103503, 2007.
- P. E. T. Jorgensen. Analysis and probability: wavelets, signals, fractals, volume 234 of Graduate Texts in Mathematics. Springer, New York, 2006.
- D. Dutkay and P. E. T. Jorgensen“Fractals on Wavelets” Revista Mathematica Iberoamericana, Vol. 22, No. 1, pp. 131-180, 2006.
- J. D’Andrea, K. Merrill and J. Packer, “Fractal Wavelets of Dutkay-Jorgensen Type for the Sierpinski Gasket Space” Contemporary Mathematics, v. 451, pp. 69-88, 2008.