Multiple tilings defined by generalized β-transformations

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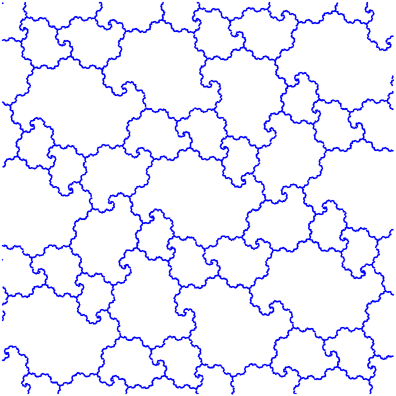
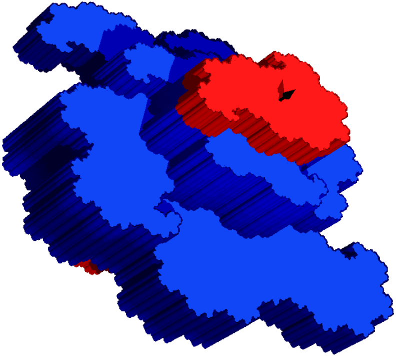
Fractals and Tilings, St. Wolfgang

Aperiodic tilings of $\mathbb{R}^{d-1}$ and lattice tilings $\{\widehat{X} + \mathbf{z}\}_{\mathbf{z} \in \mathbb{Z}^d}$

Example:





Transformations generating digital expansions in base β

- Let $\beta > 1$ be a real number.
- Let A be a finite set of real numbers, the digit set.
- For each $a \in A$, let $X_a = [\ell_a, r_a)$ be a half open interval, such that $r_a \leq \ell_b$ for $a < b$. Set $X = \bigcup_{a \in A} X_a$.
- Define the transformation $T : X \to X$ by

$$T(x) = \beta x - a \quad \text{if} \quad x \in X_a.$$

- Define the digit sequence $b(x) = b_1(x) b_2(x) \cdots$ by

$$b_k(x) = a \quad \text{if} \quad T^{k-1}(x) \in X_a.$$

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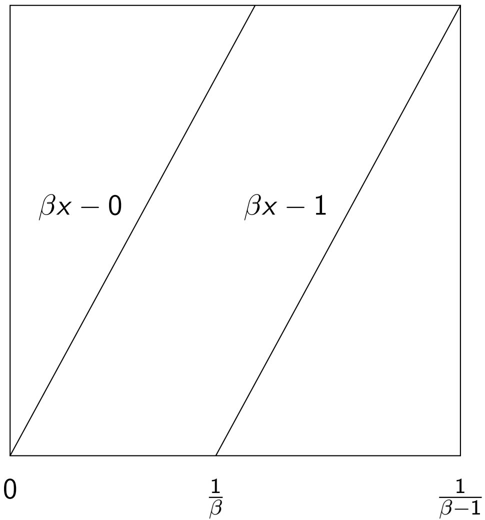
Then

$$x = \frac{b_1(x)}{\beta} + \frac{T(x)}{\beta} = \frac{b_1(x)}{\beta} + \frac{b_2(x)}{\beta^2} + \frac{T^2(x)}{\beta} = \cdots = \sum_{k=1}^{\infty} \frac{b_k(x)}{\beta^k},$$

and we call $b(x)$ the T-expansion of x.

Example: $1 < \beta \leq 2$, $A = \{0, 1\}$



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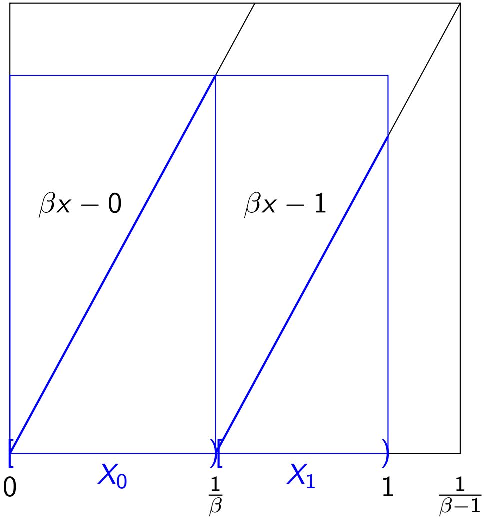

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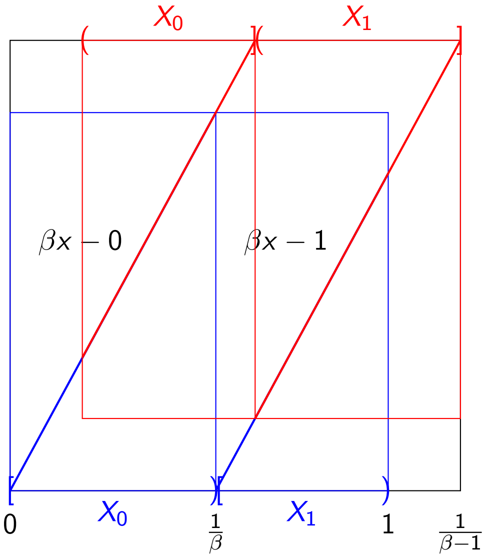

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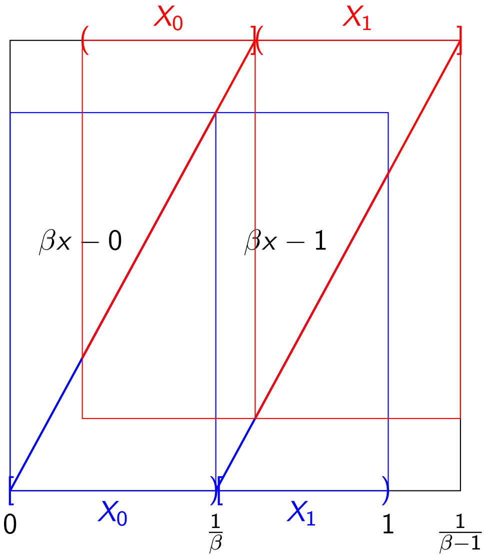

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intermediate expansions
different alphabets

Conditions on β, companion matrix, eigenvectors

Let $\beta > 1$ be a Pisot unit, i.e., an algebraic integer with minimal polynomial of the form $X^d - c_1 X^{d-1} - c_2 X^{d-2} - \cdots - c_d \in \mathbb{Z}[X]$, $|\beta_j| < 1$ for every other root $\beta_2, \ldots, \beta_d$ of the minimal polynomial, and $c_d \in \{-1, 1\}$.

Let M be the companion matrix

$$M = \begin{pmatrix} c_1 & c_2 & \cdots & c_{d-1} & c_d \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

Note that $|\det M| = |c_d| = 1$. Let $\mathbf{v}_j \in \mathbb{C}(\beta_j^{d-1}, \ldots, \beta_j, 1)^t$, $1 \le j \le d$, be a right eigenvector of M to the eigenvalue β_j (with $\beta_1 = \beta$) such that $\sum_{j=1}^d \mathbf{v}_j = \mathbf{e}_1 = (1, 0, \ldots, 0)^t$.

Tiles in the contractive hyperplane

Let H be the hyperplane of $\mathbb{R}^d$ spanned by the real and imaginary parts of $\mathbf{v}_2, \ldots, \mathbf{v}_d$. Define the map

$$\Phi : \mathbb{Q}(\beta) \to H, \quad x \mapsto \sum_{j=2}^{d} x^{(j)} \mathbf{v}_j,$$

where $x^{(j)} \in \mathbb{Q}(\beta_j)$ is defined by $(P(\beta))^{(j)} = P(\beta_j)$ for $P \in \mathbb{Q}[X]$.

Assume that $A \subset \mathbb{Q}(\beta)$. For $x \in \mathbb{Q}(\beta) \cap X$, define the tile

$$\mathcal{T}_x = \operatorname*{Lim}_{n\to\infty} \Phi\big(\beta^n T^{-n}(x)\big) = \operatorname*{Lim}_{n\to\infty} M^n \Phi\big(T^{-n}(x)\big),$$

where Lim denotes the Hausdorff limit.

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where Lim denotes the Hausdorff limit. If $y \in T^{-n}(x)$, then

$$\beta^n y = \beta^n \left(\sum_{k=1}^{n} \frac{b_k(y)}{\beta^k} + \frac{T^n(y)}{\beta^n} \right) = b_1(y)\beta^{n-1} + \cdots + b_n(y)\beta^0 + x,$$

$$\Phi(\beta^n y) = M^{n-1}\Phi\big(b_1(y)\big) + \cdots + M^0\Phi\big(b_n(y)\big) + \Phi(x).$$

Structure of the tiles $\mathcal{T}_x$

$$\mathcal{T}_x = \operatorname*{Lim}_{n\to\infty} \Phi\big(\beta^n T^{-n}(x)\big)$$
$$= \Phi(x) + \underbrace{\operatorname*{Lim}_{n\to\infty} \Big\{ M^{n-1}\Phi\big(b_1(y)\big) + \cdots + \Phi\big(b_n(y)\big) \;\Big|\; T^n(y) = x \Big\}}_{\mathcal{D}_x}$$

$$\mathcal{D}_x = \Big\{ \sum_{k=0}^{\infty} M^k \Phi(u_{-k}) \;\Big|\; \cdots u_{-1} u_0 b_1(x) b_2(x) \cdots \in \mathcal{S} \Big\},$$

where $\mathcal{S}$ denotes the set of two-sided sequences $u = (u_k)_{k\in\mathbb{Z}} \in A^{\mathbb{Z}}$ such that every suffix of u is a T-expansion of some $y \in X$:

$$\mathcal{S} = \{u \in A^{\mathbb{Z}} \mid u_k u_{k+1} \cdots \in b(X) \text{ for all } k \in \mathbb{Z}\}$$

M is contracting on H, $\{\cdots u_{-1} u_0 \mid \cdots u_{-1} u_0 b_1(x) b_2(x) \cdots \in \mathcal{S}\}$ is compact for every $x \in X \implies \mathcal{D}_x$ is compact as the image of a compact set by a continuous map

Structure of T-expansions

Define $\widetilde{X}_a = (\ell_a, r_a]$, $\widetilde{X} = \bigcup_{a\in A} \widetilde{X}_a$, $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ with

$$\widetilde{T}(x) = \beta x - a \quad \text{if} \quad x \in \widetilde{X}_a,$$

$\tilde{b}(x) = \tilde{b}_1(x)\tilde{b}_2(x)\cdots$ with

$$\tilde{b}_k(x) = a \quad \text{if} \quad \widetilde{T}^{k-1}(x) \in \widetilde{X}_a.$$

Theorem

A sequence $u = u_1 u_2 \cdots \in A^\omega$ is the T-expansion of some $x \in X$, i.e. $u = b(x)$, if and only if

$$b(\ell_{u_k}) \leq_{\mathrm{lex}} u_k u_{k+1} \cdots <_{\mathrm{lex}} \tilde{b}(r_{u_k}) \quad \textit{for all } n \geq 1.$$

The closure $\bar{\mathcal{S}}$ of $\mathcal{S}$ is a sofic shift if $b(\ell_a)$ and $\tilde{b}(r_a)$ are eventually periodic for every $a \in A$. If $\bar{\mathcal{S}}$ is a sofic shift and $T(X) = X$, then $b(\ell_a)$ and $\tilde{b}(r_a)$ are eventually periodic for every $a \in A$.

When is $\mathcal{D}_x = \mathcal{D}_y$?

It follows from the lexicographic condition for T-expansions that $\mathcal{D}_x = \mathcal{D}_y$, $x < y$, if no suffix s of a word $b(\ell_a)$ or $\tilde{b}(r_a)$, $a \in A$, satisfies $b(x) <_{\mathrm{lex}} s \leq_{\mathrm{lex}} b(y)$.

We can say more: Assume $T(X) = X$ and set

$$\mathcal{V} = \bigcup_{1\leq k<m_a,\, a\in A:\, r_a\in X} \{\widetilde{T}^k(r_a),\ T^k(r_a)\} \cup \{\ell_a \notin \widetilde{X}\} \cup \{r_a \notin X\},$$

where m_a is the minimal positive integer such that

$$\widetilde{T}^{m_a}(r_a) = T^{m_a}(r_a),$$

with $m_a = \infty$ if $\widetilde{T}^k r_a \neq T^k r_a$ for all $k \geq 1$.

Proposition

If $x < y$ and no $s \in \mathcal{V}$ satisfies $b(x) <_{\mathrm{lex}} s \leq_{\mathrm{lex}} b(y)$, then $\mathcal{D}_x = \mathcal{D}_y$.

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Idea: If $\cdots u_{-1} u_0 b(x) \in \mathcal{S}$, but $\cdots u_{-1} u_0 b(y) \notin \mathcal{S}$, then there exists an $n \ge 0$ and an $a \in A$ with $r_a \in X$ such that

- $u_{-n} \cdots u_0 = \tilde{b}_1(r_a) \cdots \tilde{b}_{n+1}(r_a)$
- $s = \tilde{b}_{n+2}(r_a) \tilde{b}_{n+3}(r_a) \cdots = \tilde{b}(\widetilde{T}^{n+1}(r_a))$
- $b(x) <_{\rm lex} s \le_{\rm lex} b(y)$.

Since $s \notin \mathcal{V}$, we have $\widetilde{T}^{n+1}(r_a) = T^{n+1}(r_a)$, thus

$$M^n \Phi\big(\tilde{b}_1(r_a)\big) + \cdots + \Phi\big(\tilde{b}_{n+1}(r_a)\big) = M^n \Phi\big(b_1(r_a)\big) + \cdots + \Phi\big(b_{n+1}(r_a)\big).$$

If $\cdots u_{-n-2} u_{-n-1} b_1(r_a) \cdots b_{n+1}(r_a) b(y) \in \mathcal{S}$, then we have shown $\sum_{k=0}^{\infty} M^k \Phi(u_{-k}) \in \mathcal{D}_y$. Otherwise, iterate with $x = T^{n+1}(r_a)$.

Consequence: The number of different sets $\mathcal{D}_x$, $x \in X$, is bounded by the number of elements in $\mathcal{V}$. (Often, it is $\#\mathcal{V} - 1$).

Remark: The set $\mathcal{V}$ is finite if and only if, for each $a \in A$ such that $r_a \in X$, $b(r_a)$ and $\tilde{b}(r_a)$ are eventually periodic or $m_a < \infty$.

Properties of $\{\mathcal{T}_x\}_{x\in\mathbb{Z}[\beta]\cap X}$ (when $A \subset \mathbb{Z}[\beta]$)

$\Phi\big(\mathbb{Z}[\beta] \cap X\big)$ is a Delone set (uniformly discrete, relatively dense).

The tiles are uniformly bounded, thus $\{\mathcal{T}_x\}_{x\in\mathbb{Z}[\beta]\cap X}$ is locally finite.

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Every tile subdivides into contracted copies of other tiles:

$$\mathcal{T}_x = \bigcup_{y\in T^{-1}(x)} M\,\mathcal{T}_y, \quad \mathcal{D}_x = \bigcup_{y\in T^{-1}(x)} \Big(M\,\mathcal{D}_y + \Phi\big(b_1(y)\big)\Big)$$

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If $\mathcal{V}$ is finite, this gives a GIFS, the union is disjoint up to a set of measure zero, $\lambda^{d-1}(\partial\mathcal{T}_x) = 0$, and $\{\mathcal{T}_x\}_{x\in\mathbb{Z}[\beta]\cap X}$ is quasi-periodic.

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If every tile has positive measure, then every tile is the closure of its interior.

Theorem

If β is a Pisot unit, $A \subset \mathbb{Z}[\beta]$, $\mathcal{V}$ is finite, then $\{\mathcal{T}_x\}_{x\in\mathbb{Z}[\beta]\cap X}$ is a multiple tiling of H. (There exists some $m \geq 1$ such that almost every point of H lies in exactly m tiles $\mathcal{T}_x$, $x \in \mathbb{Z}[\beta]\cap X$.)

cf. Thurston (1989), Praggastis (1999), Akiyama (1999, 2002), Ito–Rao (2006), Berthé–Siegel (2005)

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The tiling need not be self affine (in the sense of Praggastis (1999)): If $\bar{\mathcal{S}}$ is not sofic, then there is no partition $\{X_i\}_{i\in I}$ of X with a finite set I such that $\mathcal{D}_x$ does not change on X_i and the elements of $T^{-1}(x)$ belong to the same sets X_j for all $x \in X_i$.

Periodic T-expansions

Theorem

If β is a Pisot number and $A \subset \mathbb{Q}(\beta)$, then $b(x)$ is eventually periodic if and only if $x \in \mathbb{Q}(\beta) \cap X$.

cf. Bertrand (1977), K. Schmidt (1980), Frank–Robinson (2008)

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If β is a Pisot unit and $A \subset \mathbb{Q}(\beta)$, then
$b(x)$ is purely periodic if and only if $x \in \mathbb{Q}(\beta) \cap X$ and $\mathbf{0} \in \mathcal{T}_x$.

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Equivalently: $b(x)$ purely periodic $\Longleftrightarrow$ $x \in \mathbb{Q}(\beta)$ and $\Psi(x) \in \widehat{X}$,

$$\Psi : \mathbb{Q}(\beta) \to \mathbb{R}^d, \quad x \mapsto \sum_{j=1}^{d} x^{(j)} \mathbf{v}_j = x\mathbf{v}_1 + \Phi(x),$$

$$\widehat{X} = \bigcup_{x \in X} (x\mathbf{v}_1 - \mathcal{D}_x), \quad x\mathbf{v}_1 - \mathcal{D}_x = \Psi(x) - \mathcal{T}_x \text{ if } x \in \mathbb{Q}(\beta) \cap X$$

Remark: $x \in \mathbb{Q} \Rightarrow \Psi(x) = (x, 0, \ldots, 0)^t$

Properties of $\widehat{X}$

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Set $\widehat{X}_a = \bigcup_{x\in X_a}(x\mathbf{v}_1 - \mathcal{D}_x)$ and define

$$\widehat{T}: \widehat{X} \to \widehat{X}, \quad \widehat{T}(\mathbf{x}) = M\mathbf{x} - \Psi(a) \quad \text{if } \mathbf{x} \in \widehat{X}_a,$$
$$\pi: \widehat{X} \to X, \quad \pi(\mathbf{x}) = x \quad \text{if } \mathbf{x} = x\mathbf{v}_1 - \mathbf{y},\ \mathbf{y} \in H$$

Then $\widehat{T}$ is bijective up to a set of measure zero, π is surjective,

$$\pi \circ \widehat{T} = T \circ \pi \quad \text{and} \quad \bigvee_{k\geq 0} \widehat{T}^k(\pi^{-1}(\mathcal{B})) = \widehat{\mathcal{B}},$$

where $\mathcal{B}$ and $\widehat{\mathcal{B}}$ are the Lebesgue σ-algebras on X and $\widehat{X}$ resp.

$\Rightarrow$ $(\widehat{X}, \widehat{\mathcal{B}}, \widehat{T})$ is a natural extension of the dynamical system $(X, \mathcal{B}, T)$,
λ^d is an invariant measure of $(\widehat{X}, \widehat{\mathcal{B}}, \widehat{T})$

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Set $\widehat{X}_a = \bigcup_{x \in X_a} (x\mathbf{v}_1 - \mathcal{D}_x)$ and define

$$\widehat{T}: \widehat{X} \to \widehat{X}, \quad \widehat{T}(\mathbf{x}) = M\mathbf{x} - \Psi(a) \quad \text{if } \mathbf{x} \in \widehat{X}_a.$$

If $a \in \mathbb{Z}[\beta]$, then $\Psi(a) \in \mathbb{Z}^d$, hence

$$\widehat{T}(\mathbf{x}) \equiv M\mathbf{x} \pmod{\mathbb{Z}^d}.$$

Proposition

If $A \subset \mathbb{Z}[\beta]$ and $\mathbf{0} \in \mathcal{D}_x$, then $\beta^{k-1} x\mathbf{v}_1 \in \widehat{X}_{b_k(x)} \pmod{\mathbb{Z}^d}$.

If $\{\mathbf{z} + \widehat{X}\}_{\mathbf{z} \in \mathbb{Z}^d}$ is a tiling of $\mathbb{R}^d$ and $\mathbf{0}$ is an inner point of $\mathcal{D}_x$ for every $x \in X$, then

$$b_k(x) = a \quad \text{if and only if} \quad \beta^{k-1} x\mathbf{v}_1 \in \widehat{X}_a \pmod{\mathbb{Z}^d}.$$

Degree of the multiple tiling

Set $\varepsilon = \min_{x \in P}(r_{b_1(x)} - x)\beta$, where

$$P = \{x \in \mathbb{Z}[\beta] \cap X \mid b(x) \text{ is purely periodic}\}.$$

P is finite (since $\mathbf{0} \in \mathcal{T}_x$ for every $x \in P$), thus $\varepsilon > 0$.

Proposition

Let $z \in \mathbb{Z}[\beta] \cap [0, \infty)$ and $k \geq 0$ such that $\beta^{-k} z \in [0, \varepsilon)$. Then $\Phi(z)$ lies exactly in the tiles $\mathcal{T}_{T^k(x+\beta^{-k}z)}$, $x \in P$.

$\Phi(\mathbb{Z}[\beta] \cap [0, \infty)$ is dense in H. Consider two properties:

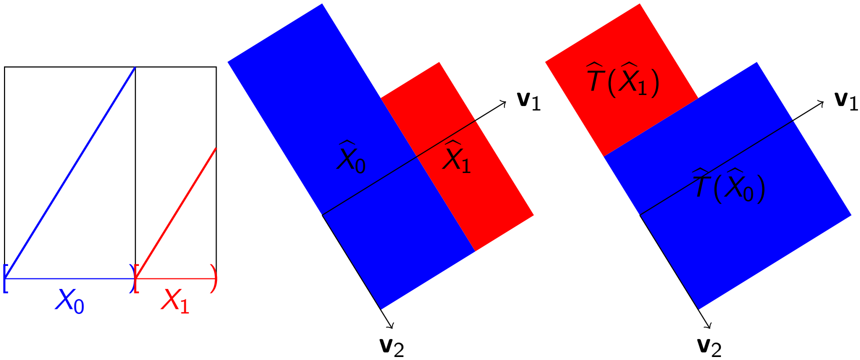
$$\begin{aligned} (F): \quad & P \text{ consists only of one element.} \\ (W): \quad & \exists\, y \in P : \forall\, x \in P\ \exists\, z \in \mathbb{Z}[\beta] \cap [0, \varepsilon),\ k \geq 0 : \\ & T^k(x+z) = T^k(y+z) = y. \end{aligned}$$

Theorem

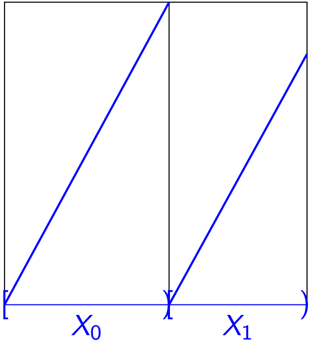
If $\{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$ is a multiple tiling, then $\{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$ is a tiling if and only if (W) holds.

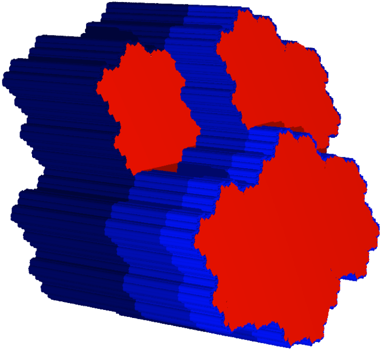
cf. Akiyama (2002); (F) $\Rightarrow$ (W)

Example: $\beta = \frac{1+\sqrt{5}}{2}$ (the golden mean), $\beta^2 = \beta + 1$, $A = \{0, 1\}$, greedy expansions.

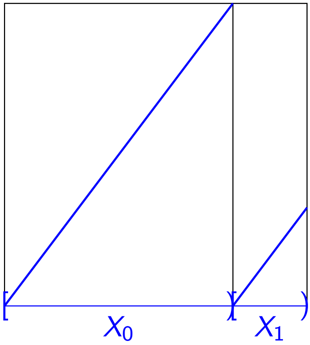


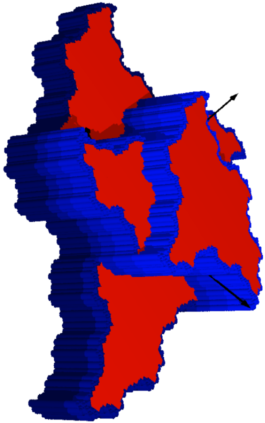
Example: $\beta^3 = \beta^2 + \beta + 1$ (Tribonacci number), $A = \{0, 1\}$, greedy expansions.





Example: $\beta^3 = \beta + 1$ (smallest Pisot number), $A = \{0, 1\}$, greedy expansions.





Example: $\beta = \frac{1+\sqrt{5}}{2}$

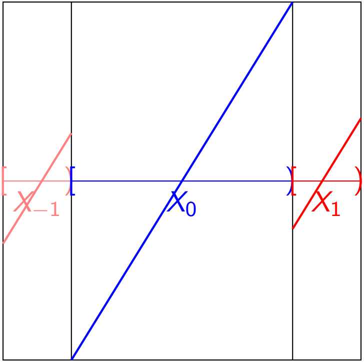
$A = \{-1, 0, 1\}$

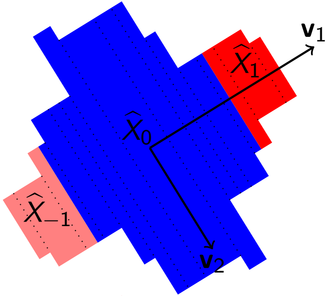
$X_{-1} = [-\alpha\,\beta, -\alpha)$, $X_0 = [-\alpha, \alpha)$

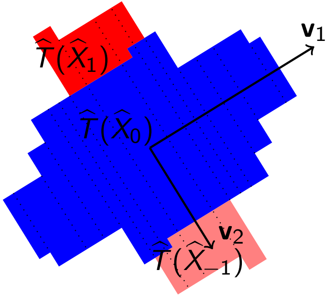
$X_1 = [\alpha, \alpha\,\beta)$, any $\alpha \in \left(\frac{\beta}{\beta^2+1}, \frac{1}{2}\right]$

$\Rightarrow \widetilde{T}^5(\pm\alpha) = T^5(\pm\alpha)$, $\#\mathcal{V} = 16$

here: $\alpha = \frac{\beta+\beta^{-4}}{\beta^2+1}$





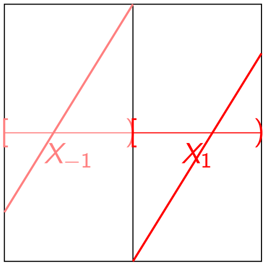


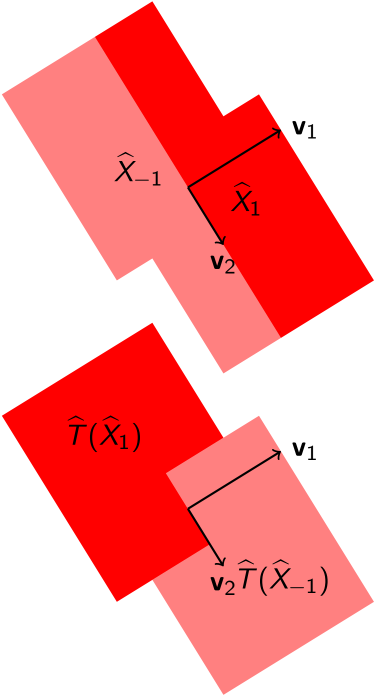
Example: $\beta = \frac{1+\sqrt{5}}{2}$

$A = \{-1, 1\}$

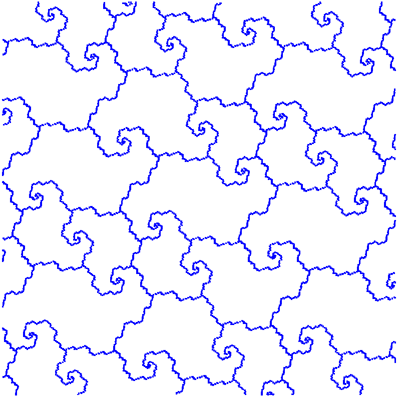
$X_{-1} = [-1, 0),\ X_1 = [0, 1)$

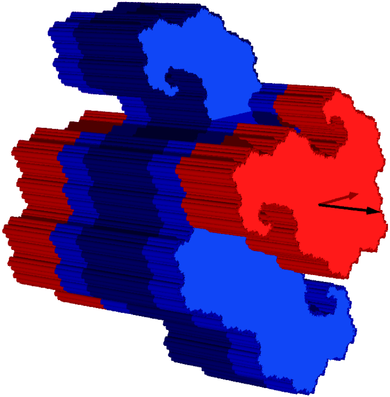
$\Rightarrow$ multiple tiling of degree 4



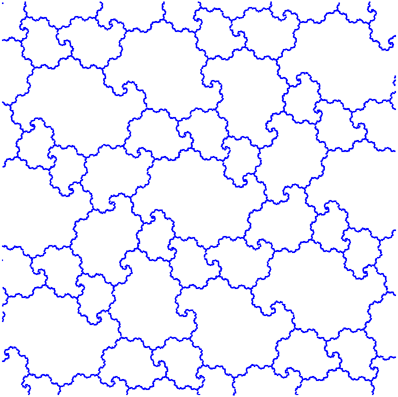


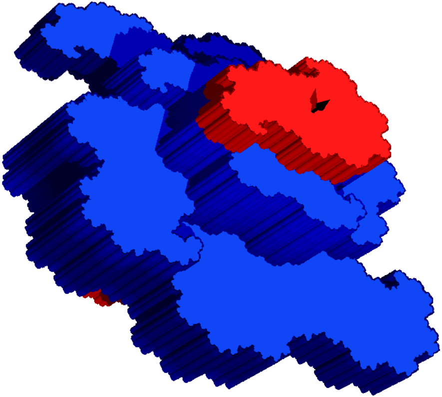
Example: $\beta^3 = \beta^2 + \beta + 1$ (Tribonacci number), $A = \{-1, 0, 1\}$, $X_{-1} = [-\alpha\,\beta, -\alpha)$, $X_0 = [-\alpha, \alpha)$, $X_1 = [\alpha, \alpha\,\beta)$, $\alpha = 1/(\beta+1)$





Example: $\beta^3 = \beta + 1$ (smallest Pisot number), $A = \{-1, 0, 1\}$,
$X_{-1} = [-\alpha\,\beta, -\alpha)$, $X_0 = [-\alpha, \alpha)$, $X_1 = [\alpha, \alpha\,\beta)$, $\alpha = \beta^2/(\beta^2 + 1)$





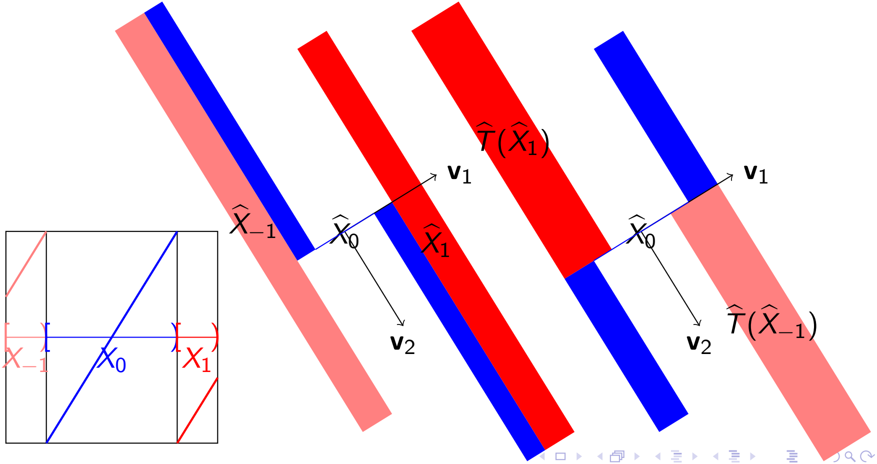
Symmetric β-transformations (Akiyama–Scheicher (2007))

$X = \left[-\frac{1}{2}, \frac{1}{2}\right),\ T(x) = \beta x - \left\lfloor \beta x + \frac{1}{2} \right\rfloor$

$\beta \le 3$: $A = \{-1, 0, 1\}$,

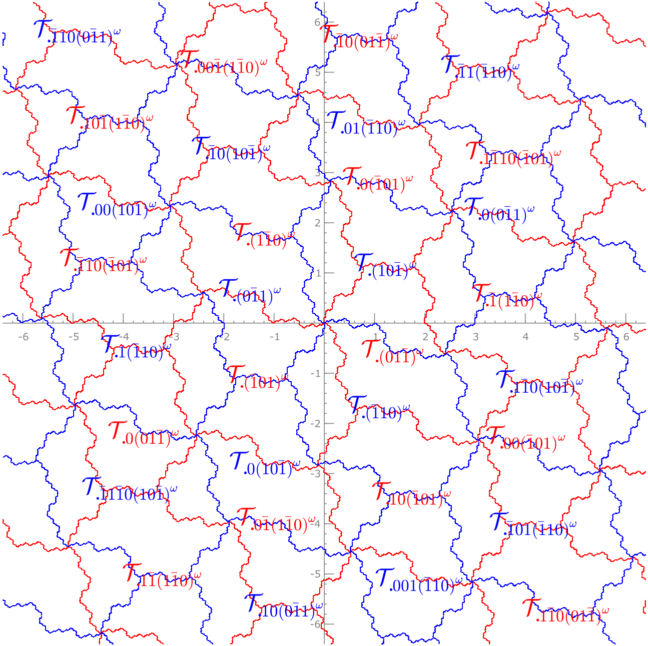
$X_{-1} = \left[-\frac{1}{2}, -\frac{1}{2\beta}\right),\ X_0 = \left[-\frac{1}{2\beta}, \frac{1}{2\beta}\right),\ X_1 = \left[\frac{1}{2\beta}, \frac{1}{2}\right)$

Example: $\beta = \frac{1+\sqrt{5}}{2}$ (the golden mean) $\Rightarrow$ tiling



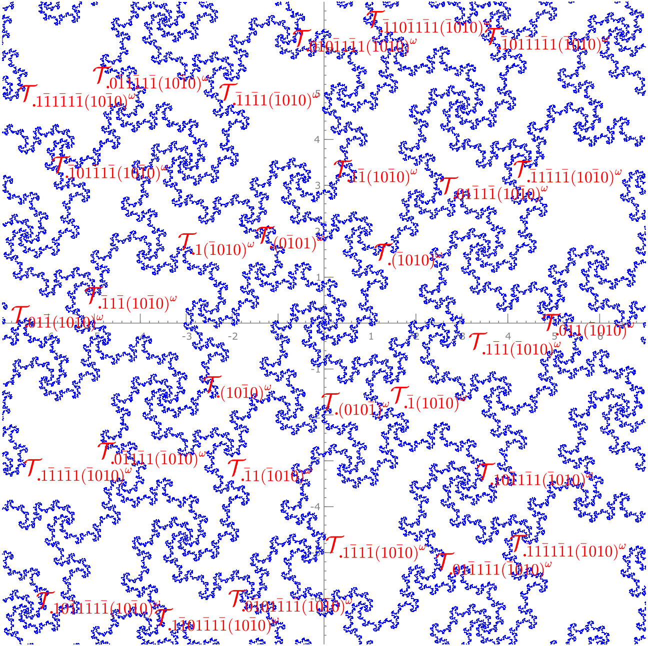
symmetric
β-transformation,
$\beta^3 = \beta^2 + \beta + 1$

$\Rightarrow$ double tiling



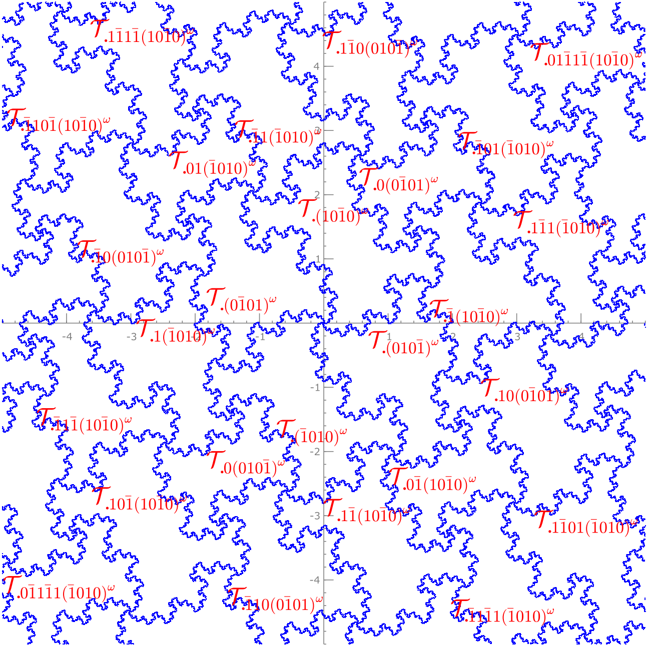
symmetric β-transformation, $\beta^3 = \beta^2 + 1$

$\Rightarrow$ tiling



symmetric β-transformation, $\beta^3 = 2\beta^2 - \beta + 1$

$\Rightarrow$ tiling



symmetric β-transformation, $\beta^3 = \beta + 1$ (smallest Pisot number)

$\Rightarrow$ double tiling