Spectrum of a Laplacian on Laakso Spaces

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8 July 2008

Laakso Spaces

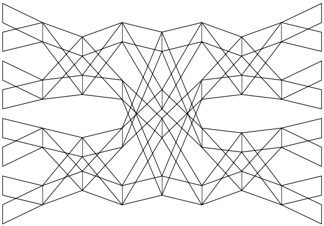

Figure: Outer Approximation

- This construction is from [Laakso].
- Quotient Space of $I \times K^k$ by "wormholes."
- Location of wormholes determined by desired dimension.
- $Dim_H = Q = 1 + k\left(\frac{\ln 2}{\ln(1/t)}\right)$.

Laakso Construction

- Fix a $Q > 1$ to be dimension.
- Let $t \geq 2$ and $k \geq 1$ be such that $Q = 1 + k\left(\frac{\ln 2}{\ln(1/t)}\right)$.
- Then if $\frac{1}{j+1} \leq \frac{1}{t} < \frac{1}{j}$ then

$$\frac{j}{j+1}\prod_{i=1}^{m} j_i^{-1} \leq t^m \leq \frac{j+1}{j}\prod_{i=1}^{m} j_i^{-1}.$$

- The $j_i \in \{j, j+1\}$ determine at each step how many subintervals to divide an interval into with the boundaries being wormhole locations.
- Identify all the wormholes and call the quotient L.

Theorem (Laakso)

With the geodesic metric and for all choices of k, j_i L is not bi-Lipschitz embedable in any $\mathbb{R}^n$.

A picture

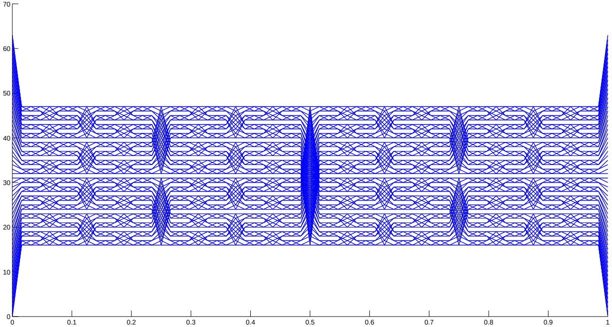

Figure: Level 6 approximation when all $j_i = 2$

Minimal Generalized Upper Gradients

Definition

On a metric measure space a minimal generalized upper gradient for a rectifiable function u is a non-negative function p_u with the following property:

$$|u(x) - u(y)| \leq \int_\gamma p_u \, dm$$

Where γ is any continuous, rectifiable path from x to y and any other function with this property is almost everywhere greater than or equal to p_u and dm is the measure induced by γ [Cheeger].

Theorem (Cheeger)

If X is a geodesic measure-metric space satisfying the Vitali covering theorem then generalized upper gradients exist for Lipschitz functions and if $1 < p < \infty$ there is a unique minimal one in L^p.

Properties of Laakso Spaces

Definition

The weak $(1,1)-$*Poincaré inequality is:*

$$\int_B |u - u_B| \, d\mu \leq C(diam(B)) \left(\int_{CB} p_u \, d\mu \right).$$

Where $B \subset L$ *is a ball,* μ *is the measure on* L*, and* C *is a constant.*

Definition

A metric measure space, X*, is* Q *Ahlfors regular if there exists two positive constants* c, c' *such that:*

$$cr^Q \leq \mu(B_r) \leq c' r^Q$$

For any $r \leq diam(X)$ *where* B_r *is a ball of radius* r *in the geodesic metric.*

Theorem (Laakso)

The space L *is a connected metric measure space which is Alfors regular of dimension* Q *supporting a* $(1,1)-$*Poincaré inequality.*

Sobolov Spaces

This definition is from [Cheeger].

Definition

The Sobolev space $H^{1,2} \subset L^2$ is defined to be $H^{1,2} = \{u \in L^2 | \exists p_u, p_u \in L^2\}$ Where p_u is the minimal upper gradient of u. $H^{1,2}$ is given the graph norm:

$$\|u\| = \left(\int u^2\right)^{1/2} + \left(\int p_u^2\right)^{1/2}.$$

Are there functions for which p_u can be hand computed?

Definition of $\mathcal{G}$

Definition

If $f \in C(L)$ it can be pulled back to a function $\hat{f}(x,w) \in C(I \times K^k)$. If a, b are any wormhole locations one of which is n'th level and the other lower level and K_n is any $n-$cell of K^k and $\hat{f}(x,w)$ is constant for fixed $x \in [a,b]$ and is continuously differentiable for fixed $w \in K_n$ then we say that $f \in \mathcal{G}_n$.
Define $\mathcal{G} = \bigcup_{n=0}^{\infty} \mathcal{G}_n$.

Lemma (S)

If $f \in \mathcal{G}_n$ then $\widehat{p_f} = \left|\frac{\partial f}{\partial x}\right|$. And this is genuinely the pull back of a function from $L^2(L)$.

A Dirichlet form

Definition

For $f \in \overline{\mathcal{G}}$ define

$$\mathcal{E}(f,f) = \int_L p_f^2 \, d\mu.$$

Theorem (S)

The form $(\mathcal{E}, \overline{\mathcal{G}})$ is a local, regular Dirichlet form.

This is proved by showing the existence of a self-adjoint non-positive definite operator, A such that $\left(\left|\frac{\partial}{\partial x}\right|\right)^2 f = -Af$ and then setting

$$\mathcal{E}(f,g) = \int_L (\sqrt{-A}f)(\sqrt{-A}g) \, d\mu.$$

Inverse Limit Systems

Definition

An inverse system of topological spaces is a family of topological spaces $\{F_i\}$ along with a family of continuous projection maps $\phi_i : F_{i+1} \to F_i$. The system can also have a family of compatible measures μ_i such that if A is μ_i measurable then $\mu_i(A) = \mu_{i+1}(\phi_i^{-1}(A))$.

Definition

The inverse limit of an inverse system is a subset of $\prod_{i=0}^{\infty} F_i$ such that there exist continuous maps $\Phi_n : \lim_{\leftarrow} F_i \to F_n$ such that $\Phi_n = \phi_{n+1} \circ \Phi_{n+1}$.

Inverse Limit Systems 2

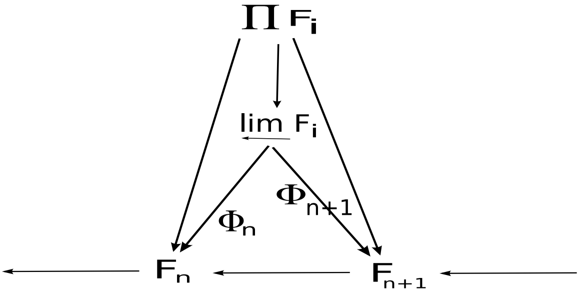

Figure: Diagram of an Inverse System

Inverse Limit Systems 3

Theorem

There exists a unique topological space that is the inverse limit of the F_i. If the masses of μ_i are bounded then there is also a unique measure μ_∞ on the limit space $\lim_{\leftarrow} F_i$.

Let $C(F_i)$ be the continuous functions on F_i then then can be pulled back onto $\lim_{\leftarrow} F_i$ using Φ_i. So $f \in C(F_i)$ gets mapped to $f \circ \Phi_i^{-1} \in C(\lim_{\leftarrow} F_i)$. This can be done for any family of function spaces on F_i and get a function space on the limit space as well.

Also families of operators can be constructed on the $\{F_i\}$ with the same property as the measures, the operators have a limit on the limit space as well.

Vermiculated Spaces

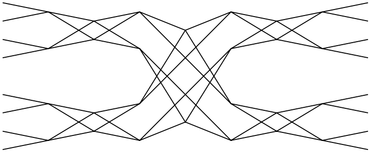

Figure: Level 3, $j_i = 2$ Approximation

- From [Barlow-Evans]
- Projective limit of quantum graphs
- Location of vertices determined by the sequence $\{j_i\}$.
- Construction includes Markov Processes.

Theorem (Barlow-Evans)

This fractal is connected, compact, metric- measure space.

Vermiculated Graphs

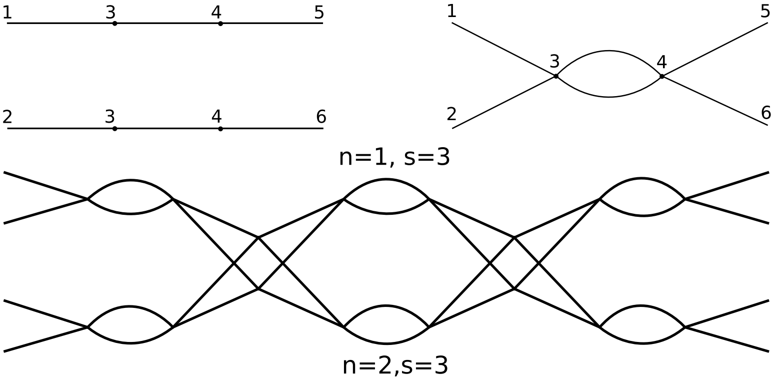

Figure: Vermiculated Approximations to a Laakso space

Construction of Vermiculated Spaces

- Assume $1 < Q < 2$ for simplicity.
- Start with a base space $F_0 = [0, 1]$. An indexing set $G_n = \{0, 1\}$, and locations B_n.
- Form $F_0 \times G_1$ and identify at the points specified in B_n, these are the same locations as the level one wormholes for Laakso.
- This gives $\{F_i\}$ and continuous projections from $F_n \to F_{n-1}$. This is a projective system which has a limit $\lim_{\leftarrow} F_i$.
- Since each F_i has a probability measure the measures limit uniquely to some probability measure μ_∞ with nice compatibility with projections back onto the F_i.

Markov Processes on Vermiculated Spaces

- Begin with a process on F_0, say X^0 this can be pieced together to form processes on the F_i, say X^i.

Theorem (Barlow-Evans)

Given mild assumptions $\exists$ a MP, X, on $\lim_{\leftarrow} F_i$ built from X^0 on F_0.

Lemma (S)

In the Laakso space construction the process X is symmetric if and only if the process X^0 is symmetric.

By the general theory of Dirichlet forms and Markov processes, this process X_t is associated to some Dirichlet form on L.

Equivalence of Constructions I

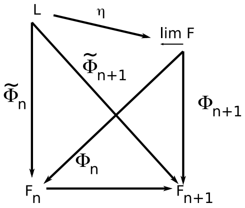

Figure: Use of the Universal Property

- Projective limits like tensor products have a universal property.
- $\tilde{\Phi}_n : L \longrightarrow F_n$ by collapsing the cantor sets down to only 2^n points are continuous.
- By Univ Prop $\exists! \eta : L \longrightarrow \lim_{\leftarrow} F_i$.

Equivalence of Constructions II

Lemma

η is a homeomorphism, so $L = \lim_{\leftarrow} F_i$ topologically.

Lemma

η is an measure preserving isometry.

Theorem (S)

The space from the Laakso construction L and $\lim_{\leftarrow} F_i$ from the vermiculated construction can be used interchangeably and so can functions spaces on them by identifying elements of the spaces through η.

Theorem (S)

The Dirichlet forms and Markov processes constructed in the Laakso and the vermiculated construction with Brownian motion on the interval coincide. That is, same generator with same domain.

Approaching the Spectrum of A

- Projective limit construction much better suited to this than Laakso's quotient construction.
- On the $n'th$ approximating quantum graph the process X_t^i has infinitesimal generator of negative twice differentiation on each line segment, so eigenfunctions and eigenvalues are easy to compute

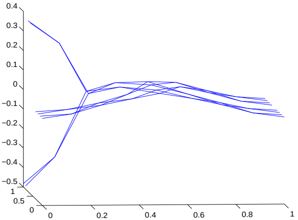

Figure: $j_i = 2, n = 3$: An eigenfunction with eigenvalue of $4\pi^2$

The Spectrum of A

The idea: A dense set of $\overline{\mathcal{G}}$ are functions pulled back to L from the approximating quantum graphs.

Definition

Let $\mathcal{D}_n \subset Dom(A) \subset \overline{\mathcal{G}}$ be those functions in $Dom(A)$ that are pull backs of functions on the $n'th$ approximating graph but not a pull back of a function on the $n-1'st$ graph.

- $\mathcal{D}_n$ are disjoint.
- $\bigcup_{n=0}^{\infty} \mathcal{D}_n$ is dense in $Dom(A)$.

$$d_n = \prod_{i=1}^{n} j_i^1$$

$$\sigma(A) = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{\infty} \left\{ \frac{k^2\pi^2}{d_n^2} \right\} \cup \bigcup_{n=2}^{\infty} \bigcup_{k=1}^{\infty} \left\{ \frac{k^2\pi^2}{4d_n^2} \right\} \cup \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{\infty} \left\{ \frac{(2k+1)^2\pi^2}{4d_n^2} \right\}$$

- M. Barlow and S. Evans. Markov processes on vermiculated spaces. *Random walks and geometry*, (ed. V. Kaimanovich), de Gruyter, Berlin, 2004.
- J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, *Geom. Funct. Anal.* **9** (1999), no. 3, 428–517.
- T.J. Laakso. Ahlfors $Q-$regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality. *Geom. Funct. Anal.*, **10** (2000) no. 1 , 111-123.
- K. Romeo, S., *Eigenmodes of a Laplacian on some Laakso Spaces*, Complex Equations and Elliptic Equations, **54**:6, pp. 623-637 (2009).
- S., *Dirichlet Forms on Laakso and Barlow Evans Fractals of Arbitrary Dimension* **arXiv:0811:1378v2**.