Structure of planar self-affine tilings

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(Joint work with Tao Jiang and Sze-Man Ngai.)

1. Background

$A \in M(2,\mathbb{Z})$ expanding. $\mathcal{D}$ a complete set of coset representatives of $\mathbb{Z}^2/A\mathbb{Z}^2$. $T = T(A,\mathcal{D})$ satisfies $T = \bigcup_{d\in\mathcal{D}} A^{-1}(T + d)$.

Theorem A. (Bandt-Gelbrich 94, Bandt-Wang 01) In a planar lattice tiling by a disklike self affine tile T, T either has

(a) 6 edge neighbors, or

(b) 8 neighbors with 4 vertex neighbors and 4 edge neighbors.

Under suitable connectivity conditions, the converse is true.

Theorem B. (Leung-Lau 07) Let $T = T(A,\mathcal{D})$ with $A \in M(2,\mathbb{Z})$ expanding, and $\mathcal{D} = \{0, v, \ldots, (|\det A| - 1)v\}$ consecutive collinear, where $v \in \mathbb{Z}^2$, v, Av independent. Let $f(x) = x^2 + px + q$ be the characteristic polynomial of A. Then

(a) T is dislike if and only if $2|p| \le |q + 2|$.

(b) If T is disklike, T is a 'square tile' if and only if $p = 0$.

(The special case of tilings by fundamental domains of CNS: Akiyama-Thuswaldner 04.)

Question. What is the structure of the tiling generated by a non disklike T? Three aspects:

(a) the number and position of its neighbors;

(b) the way T intersects with its neighbors;

(c) describe these in terms of the characteristic polynomial of A.

(Special case: for non-disklike T with $\mathcal{D}$ consecutive collinear.)

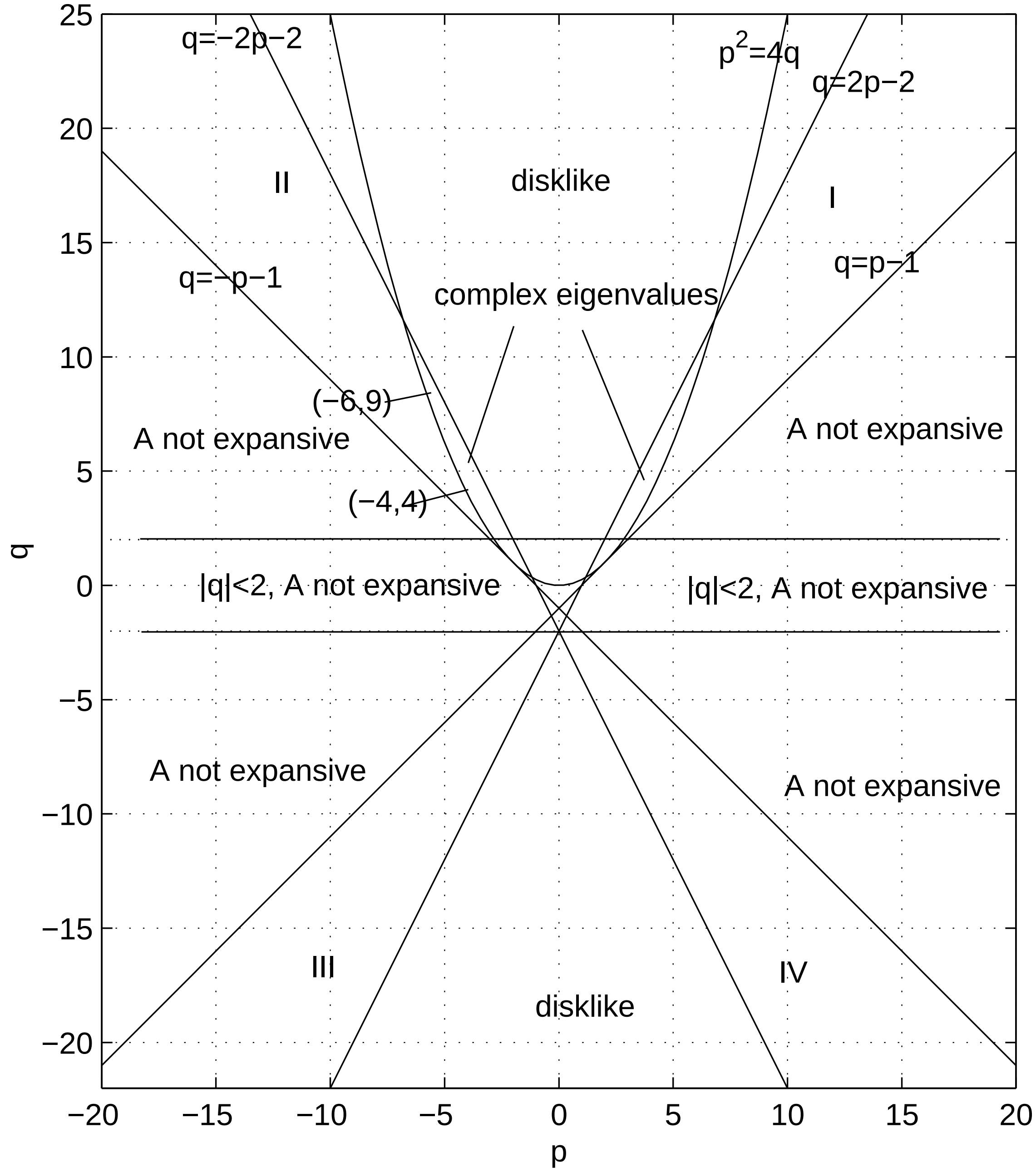

FIGURE 1. The relevant regions.

2. Simplifications

Lemma 1. $T(A,\mathcal{D})$ as above and the tiling it generates is affine equivalent to $T(C,\mathcal{E})$ and its tiling, where

$$C = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}, \quad \mathcal{E} = \{d_i = (i,0)^t, i = 0,\ldots,|q|-1\}.$$

Lemma 2. Let $T = T(A,\mathcal{D})$, $\tilde{T} = T(-A,\mathcal{D})$. Then the tilings they generate are affine equivalent.

Can assume:

$$A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}, \quad \mathcal{D} = \{d_i = (i,0)^t, i = 0,\ldots,|q|-1\}.$$

4 cases. Let ν_1, ν_2 be the eigenvalues of A^{-1}.

1. $p^2 > 4q$, $q > 0$, $p < 0$. ($\Leftrightarrow 1 > \nu_1 > \nu_2 > 0$.)

2. $p^2 > 4q$, $q < 0$, $p > 0$. ($\Leftrightarrow 1 > \nu_1 > 0 > \nu_2 > -1$.)

3. $p^2 < 4q$. Complex eigenvalues.
$A = [0,1;-3,\pm 3], [0,1;-5,\pm 4], [0,1;-7,\pm 5]$.

4. $p^2 = 4q$. $1 > \nu_1 = \nu_2 > 0$, one independent eigenvector. (2 independent eigenvectors $\Rightarrow$ disklike). $A = [0,1;-4,\pm 4], [0,1;-9,\pm 6]$.

Cases 1 and 4 can be considered simultaneously.

Definitions

(a) $T^{(m,n)} = T + (m,n)^t$.

(b) $T^{(m,n)}_{i_1\cdots i_\ell}$, $i_j \in \{0,\ldots,|q|-1\}$ are the ℓ-level pieces of $T^{(m,n)}$.

(c) $T^{(m,n)} = T + (m,n)^t$ is in the k-th layer if $(m,n) = (k,0) + l(-p,q)$.

Two tiles are in the same layer if their first level pieces are aligned along the same line of direction $A^{-1}d_1$.

(d) T and T' are *vertex neighbors* if $T \cap T'$ is finite;

(e) *Cantor neighbors* if $T \cap T'$ is a totally disconnected perfect set, possibly adjoining a countable set, resulting in a totally disconnected closed set;

(f) *edge neighbors* if $T \cap T'$ contains a compact connected set of more than one point.

3. Results

Theorem 1. Let $T = T(A, \mathcal{D})$, $A = [0, 1; -q, -p]$, $q > 0$, $p < 0$, $\mathcal{D} = \{d_i = (i, 0)^t, i = 0, \ldots, q-1\}$. Let

$$K = \left\lfloor \frac{q-1}{p+q+1} \right\rfloor .$$

Then

(a) T has $4K+2$ neighbors, 2 in each of the $2K+1$ layers.

(b) For $k = 1, \ldots K$, the neighbors in the k-th layer are $T^{(k,0)-k(-p,q)}$ and $T^{(k,0)-(k-1)(-p,q)}$. Those in the -k-th layer are $T^{-(k,0)+k(-p,q)}$ and $T^{-(k,0)+(k-1)(-p,q)}$. The ones in the *0*-th layer are $T^{\pm(-p,q)}$.

(c) If $K = (q-1)/(p+q+1)$, the K, -K-th layer neighbors are vertex neighbors. If $K < (q-1)/(p+q+1)$, they are Cantor neighbors.

(d) For $k = 2, \ldots, K-1$, the k-type intersection is the union of the miniatures of k-type and *k+1*-type intersections. The neighbors in these layers are Cantor neighbors.

(e) The neighbors in the $-1, 0, 1$-st layers are edge neighbors.

Remark: Together with the next two theorems,

(a) there are no neighbors with countable intersection;

(b) these are all the possible neighbor types.

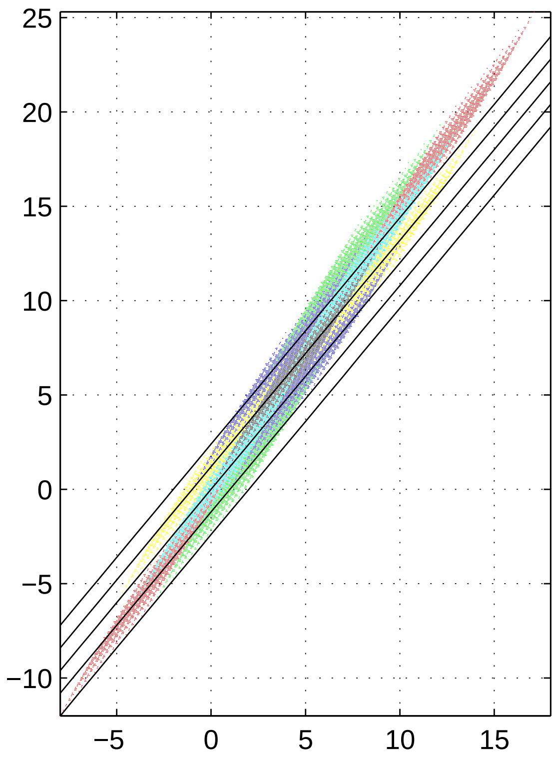

FIGURE 2. Layers of neighbors.

(a) Layers of neighbors.

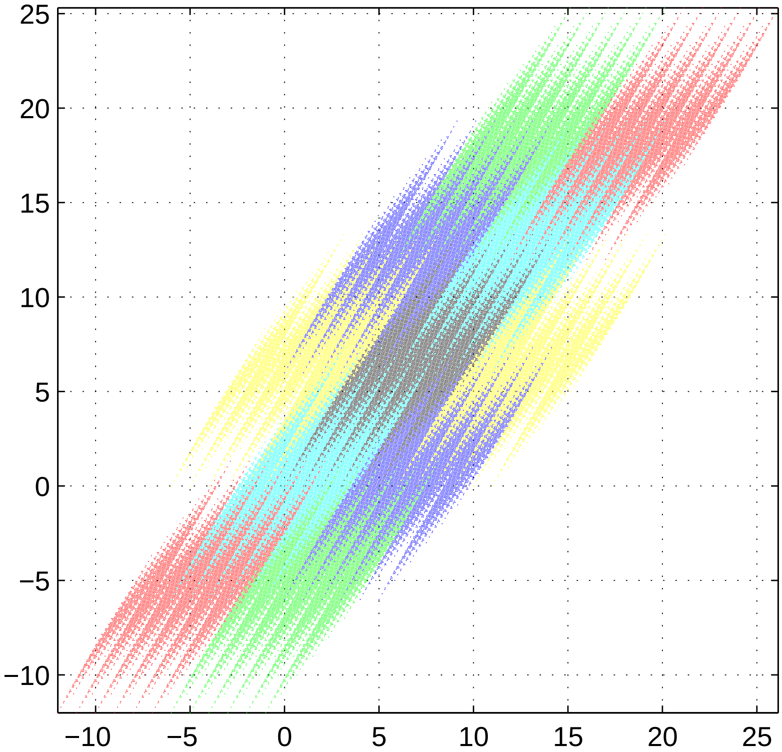

FIGURE 3. The magnified neighbors.

(b) magnified neighbors.

Theorem 2. Let $T = T(A, \mathcal{D})$, $A = [0, 1; -q, -p]$, $q < 0$, $p > 0$, $\mathcal{D} = \{d_i = (i, 0)^t, i = 0, \ldots, q-1\}$. Let

$$K = \left\lfloor \frac{|q| - 1}{|q| - p - 1} \right\rfloor .$$

Then

(a) T has $4K + 2$ neighbors, 2 in each of the $2K + 1$ layers.

(b) For $k = 1, \ldots K$, the neighbors in the -*k*-th layer are $T^{(-k,0)-k(p,-q)}$ and $T^{-(k,0)-(k-1)(p,-q)}$. The ones in the *k*-th layers are $T^{(k,0)+k(p,-q)}$ and $T^{(k,0)+(k-1)(p,-q)}$. The ones in the *0*-th layer are $T^{\pm(p,-q)}$.

(c) If $K = (|q|-1)/(|q|-p-1)$, the *K*, -*K*-th layer neighbors are vertex neighbors. If $K < (|q| - 1)/(|q| - p - 1)$, they are Cantor neighbors.

(d) For $k = 2, \ldots, K-1$, the -*k*-type intersection is the union of the miniatures of -*k*-type and -*(k+1)*-type intersections. The neighbors in these layers are Cantor neighbors.

(e) The neighbors in the $-1, 0, 1$-st layers are edge neighbors.

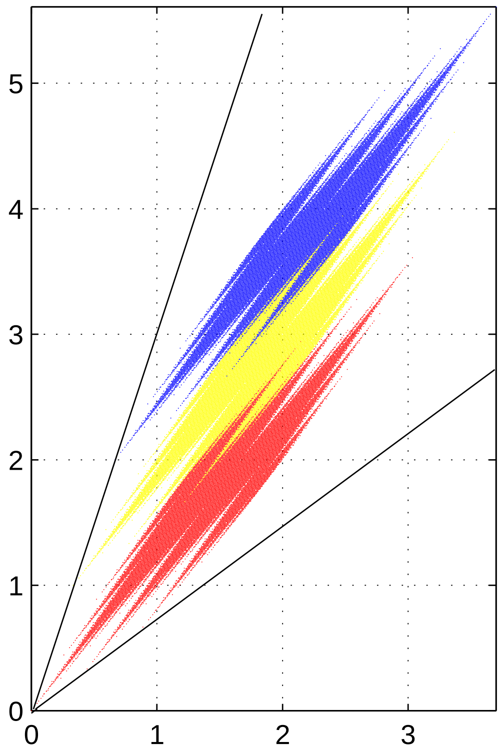

FIGURE 4. A tile in this flip-over case.

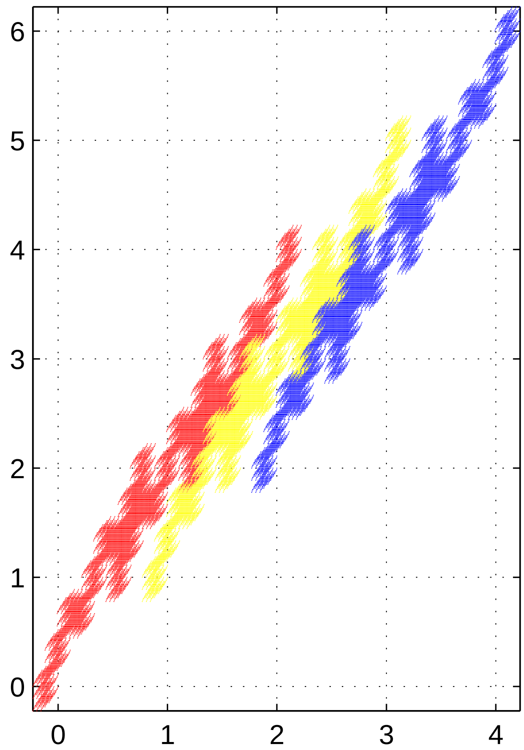

FIGURE 5. The tile with $A = [0, 1; -3, 3]$

Theorem 3. Let $T = T(A, \mathcal{D})$, $A = [0, 1; -3, \pm 3]$, $[0, 1; -5, \pm 4]$, $[0, 1; -7, \pm 5]$, $\mathcal{D} = \{d_i = (i, 0)^t, i = 0, \ldots, |\det A| - 1\}$. Then

(a) T has 10 neighbors, 2 in each of 5 layers.

(b) There are 2 vertex neighbors in each of the *2*, -*2*-nd layers.

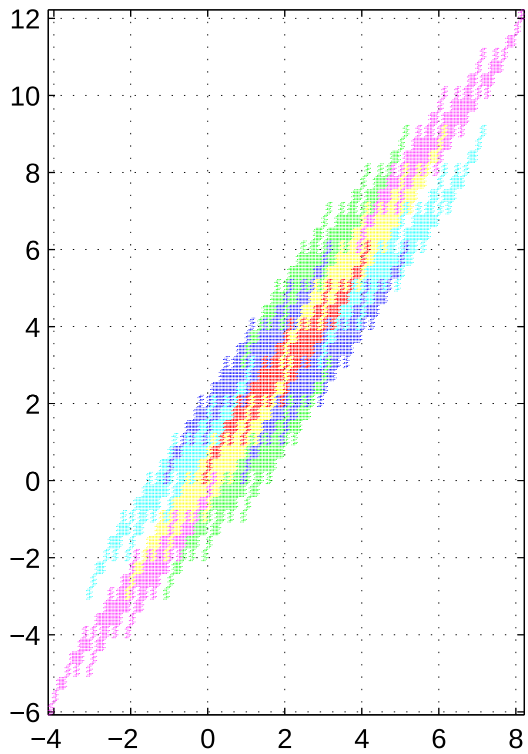

Figure 6. The neighbors of the exceptional tile.

4. Proof of Theorem 1.

Step 1.

Lemma. (a) $T(A, \mathcal{D})$ is contained in the bounding parallelogram P with vertices

$$p_0 = (0,0)$$
$$p_1 = \left(\frac{2q(q-1)}{(p^2+p\sqrt{p^2-4q}-2q)(p+q+1)}, \frac{-(p+\sqrt{p^2-4q})q(q-1)}{(p^2+p\sqrt{p^2-4q}-2q)(p+q+1)}\right)$$
$$p_2 = (q-1)(A-I)^{-1}d_1 = \left(\frac{(-p-1)(q-1)}{p+q+1}, \frac{q(q-1)}{p+q+1}\right)$$
$$p_3 = p_2 - p_1.$$

(b) $T(A, \mathcal{D})$ is contained in the bounding hexagon H with vertices

$$p_0 = (0,0)$$
$$p_4 = A^{-1}p_1$$
$$p_5 = A^{-1}p_2 = p_2 - p_7 = (q-1)(A-I)^{-1}d_1 - (q-1)A^{-1}d_1$$
$$p_2 = (q-1)(A-I)^{-1}d_1$$
$$p_6 = p_2 - p_4 = (q-1)(A-I)^{-1}d_1 - A^{-1}p_1$$
$$p_7 = (q-1)A^{-1}d_1.$$

H is the convex hull of the first level approximation of T, $\mathcal{F}^1(P) = \cup_{i=0}^{q-1} P_i$.

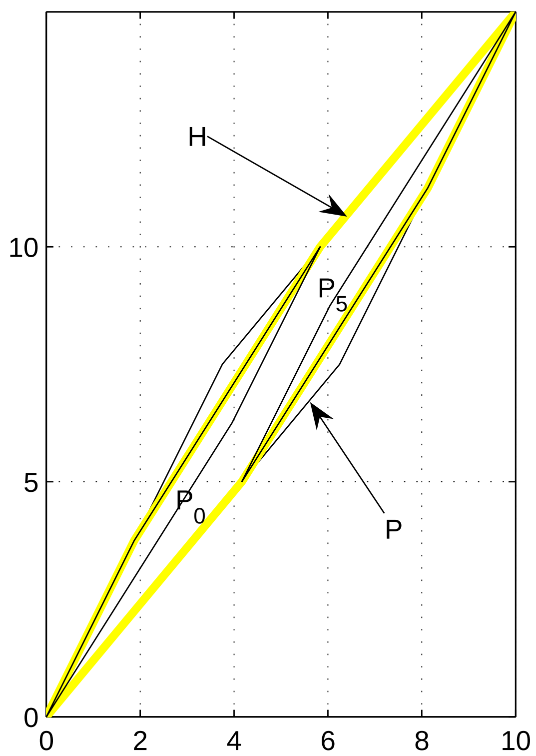

FIGURE 7. The bounding parallelogram and hexagon for $A = [0, 1; -6, 5]$.

Reason:

- From the approximations $\mathcal{F}^k(\{0\})$ of T, T is in the acute cone in the 1st quadrant bounded by the directions of $A^{-1}d_1$ and v_1, where $A^{-1}v_1 = \lambda_1 v_1$.
- Center of symmetry of T: $z = \frac{1}{2}(q-1)(A-I)^{-1}d_1$ (Duda 07). Hence T is inside the parallelogram.
- The bounding hexagon H is the convex hull of the *1*-level approximation $\mathcal{F}^1(P)$ of T.

Remark $0 \in T \Rightarrow$ the tip $p_2 = (q-1)(A-I)^{-1}d_1$ is in T. So are its iterated images.

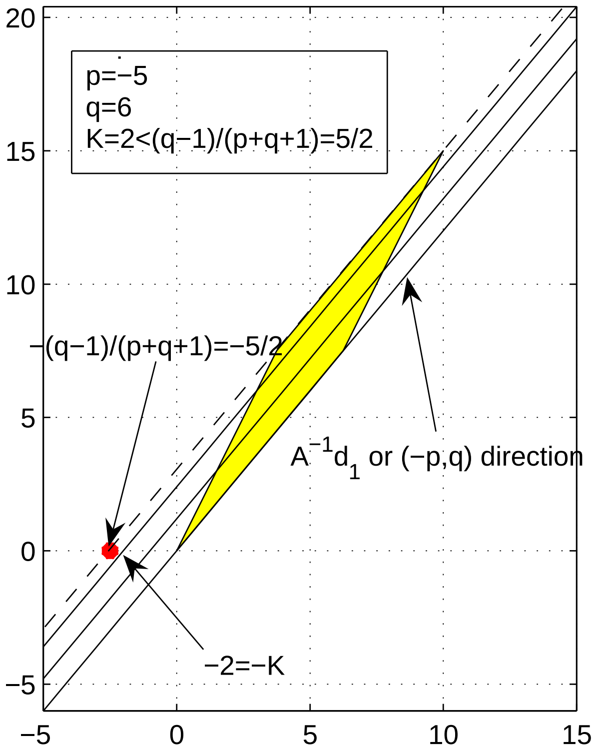

FIGURE 8. Number of layers: $A = [0, 1; -6, 5]$.

Step 2.

Claim. There are at most $2K + 1$ layers that contain a neighbor of T.

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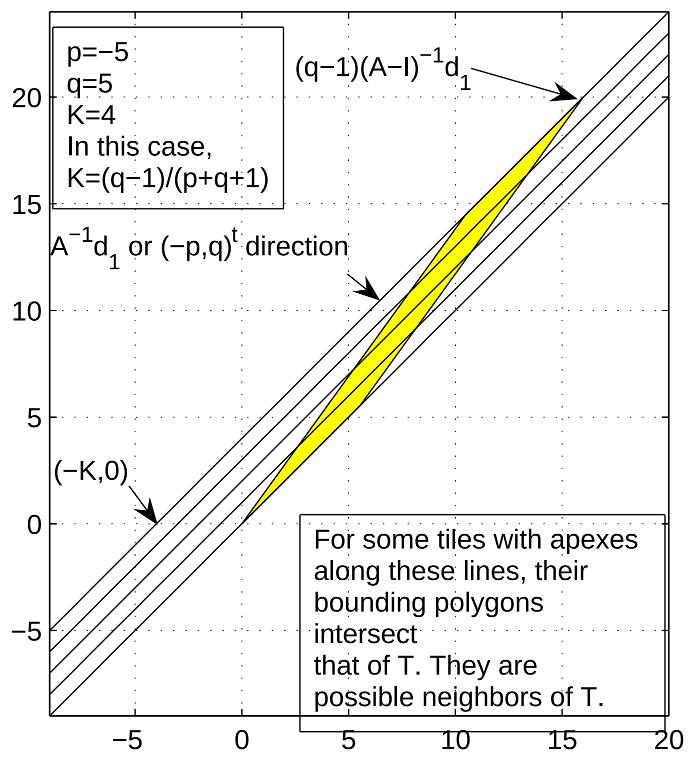

FIGURE 9. Number of layers: $A = [0, 1; -5, 5]$.

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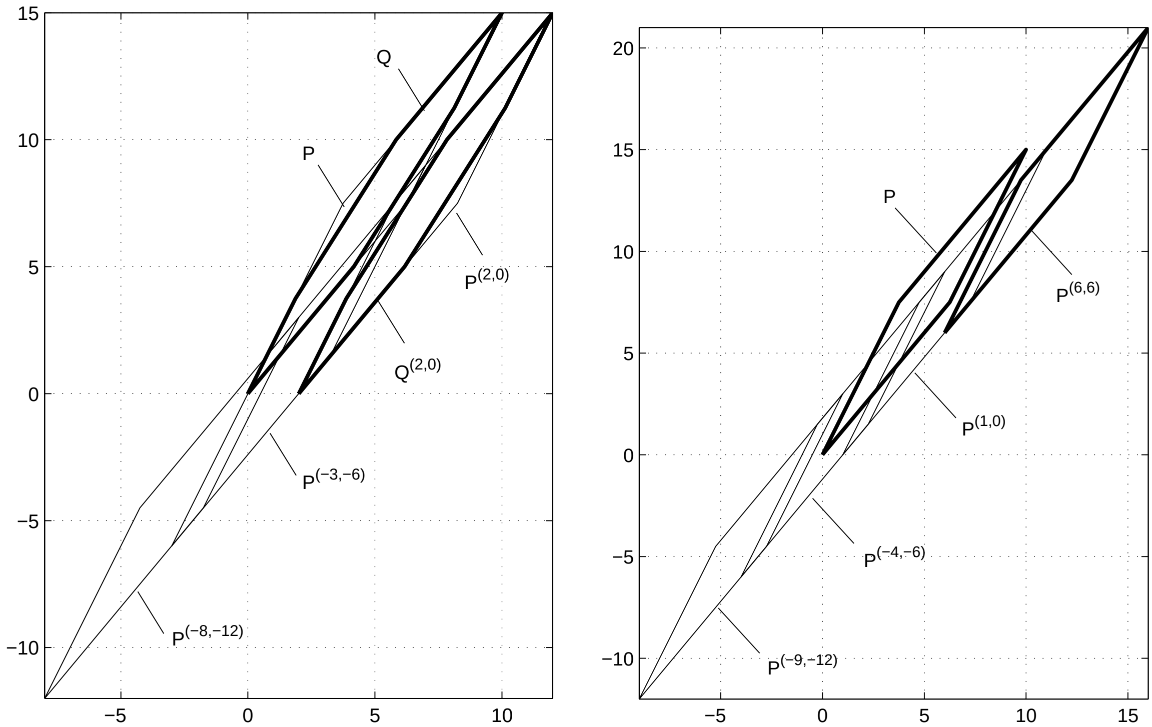

(a) The tile T.

(b) T with two vertical neighbors.

FIGURE 10. (a) number of 2nd layer neighbors is at most 2. (b) same for 1st layer neighbors.

Step 3.

Claim. There are at most 2 neighbors of T in each of the layers.

Reason: Use bounding parallelograms and bounding hexagons to separate the non-neighbors from T.

Step 4.

Claim. For $k = 1, \ldots, K$, the tiles $T^{(k,0)-k(-p,q)}$ and $T^{(k,0)-(k-1)(-p,q)}$ are the *k*-layer neighbors of T. The *0*-layer neighbors are $T^{(-p,q)}$ and $T^{(p,-q)}$.

Reason: Use radix expansions to show that the indicated tiles has common points with T.

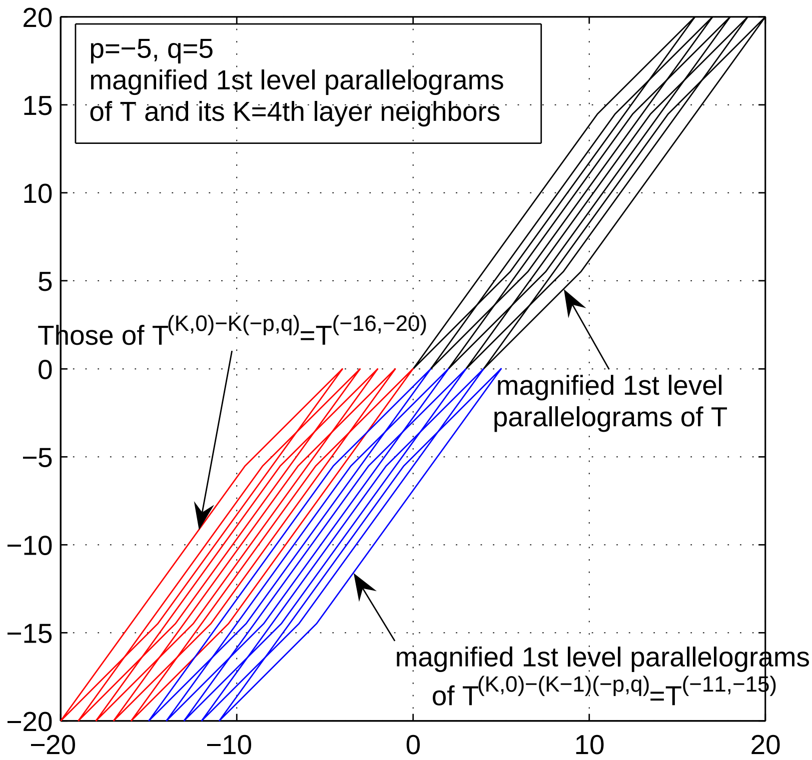

FIGURE 11. Vertex neighbors. $A = [0, 1; -5, 5]$.

Step 5.

Claim. If $K = \frac{q-1}{p+q+1}$, the K, -K-layer neighbors are vertex neighbors.

Reason: Calculate. The tip of $T^{(K,0)-K(-p,q)}$ is 0, and hence is a vertex neighbor. Neighbors in the same layer are of the same type. By symmetry, also for the -K-layer.

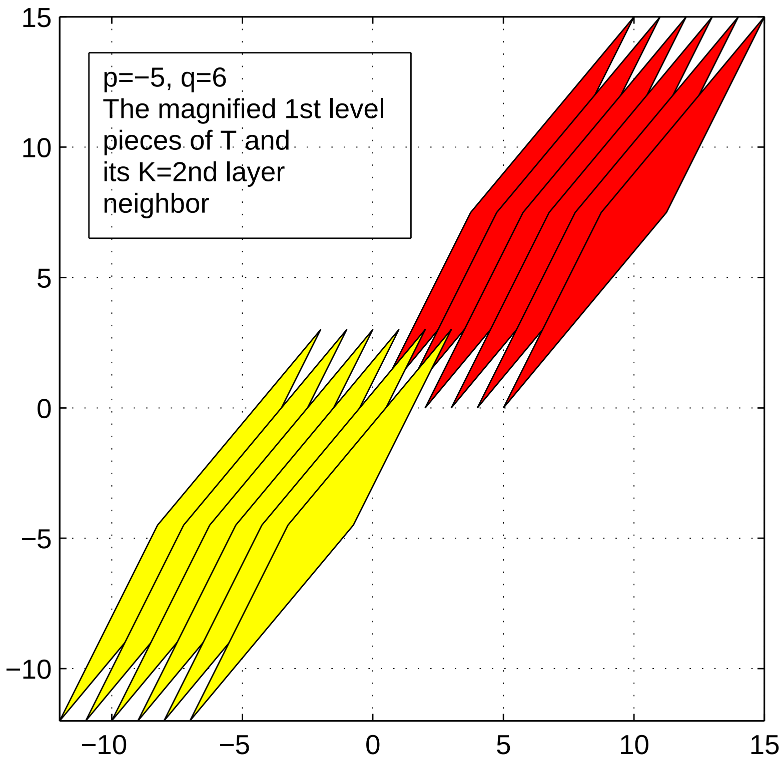

FIGURE 12. The intersection of the 1st level approximations.

Claim. If $K < \frac{q-1}{p+q+1}$, the K, -K-layer neighbors are Cantor neighbors.

Reason: Apply A^{-1} to the above picture to see what's happening at the 2nd level.

Use induction.

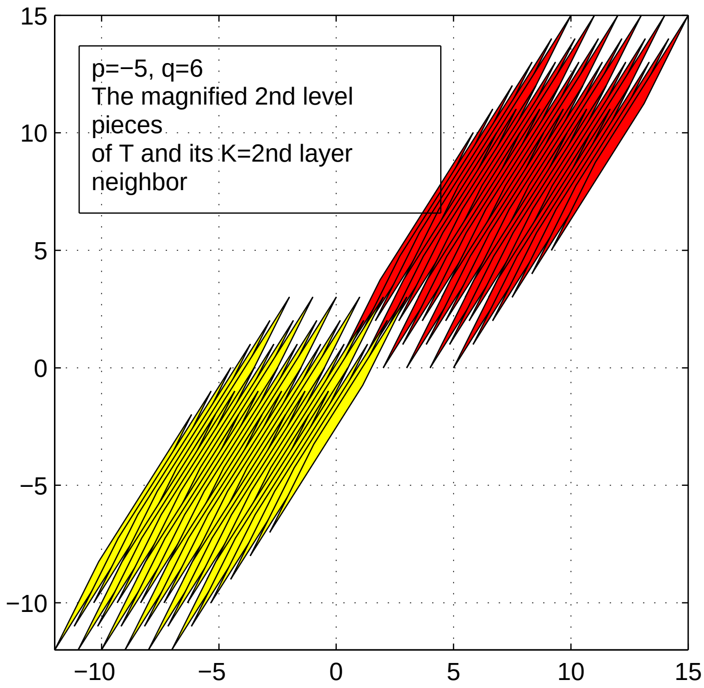

FIGURE 13. The intersection of the 2nd level approximations.

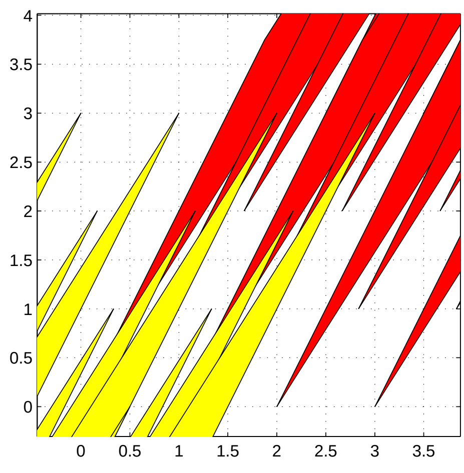

FIGURE 14. Zoom in.

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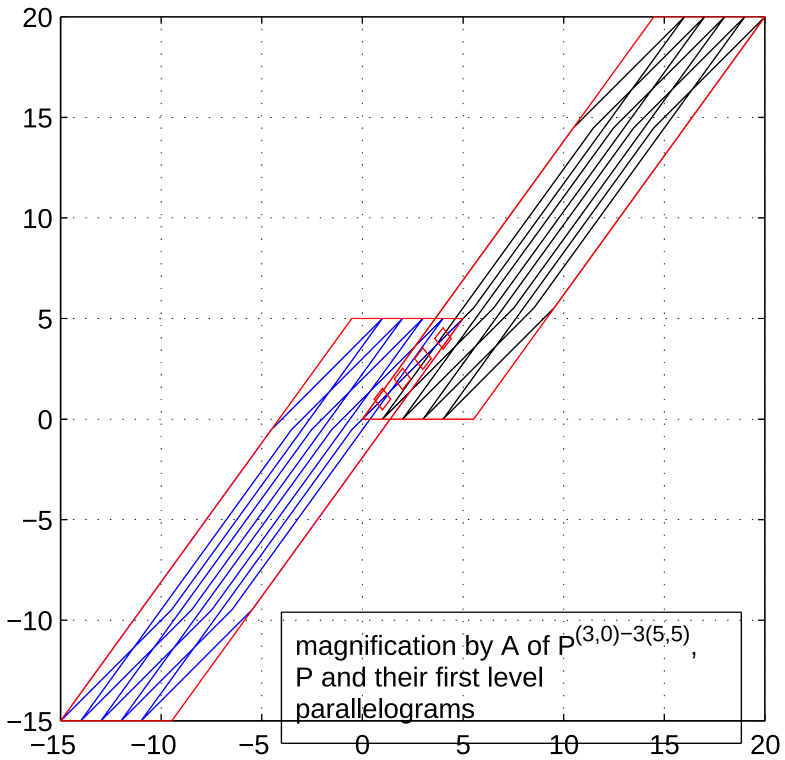

FIGURE 15. $A = [0, 1; -5, 5]$. The 3rd layer neighbor is a Cantor neighbor.

Step 6.

Claim. For $k = 2, \ldots, K - 1$, the k-type intersection is the union of the miniatures of k-type and k+*1*-type intersections. The neighbors in these layers are Cantor neighbors.

Reason: The next level in the region of k-type intersection can be seen by applying A^{-1} to the picture.

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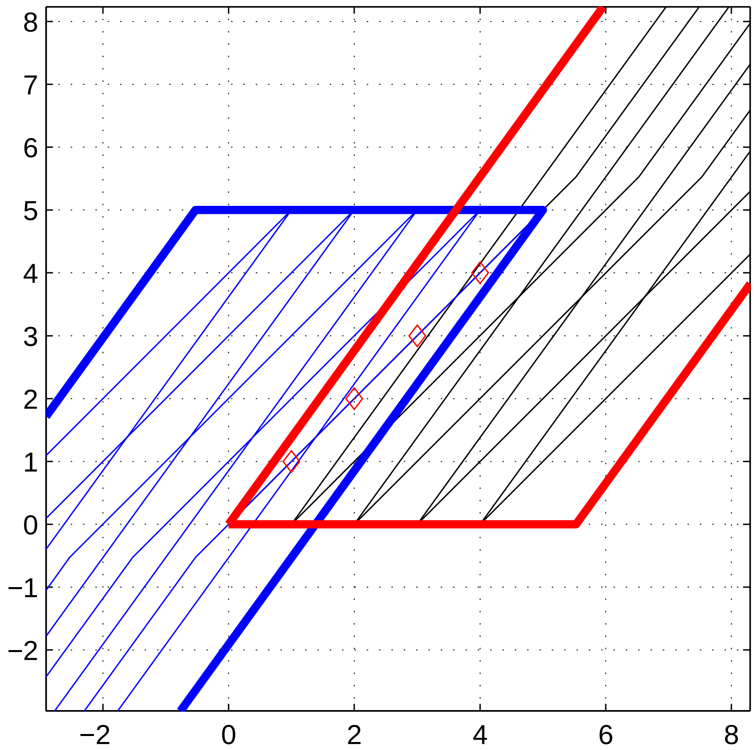

FIGURE 16. Zoom.

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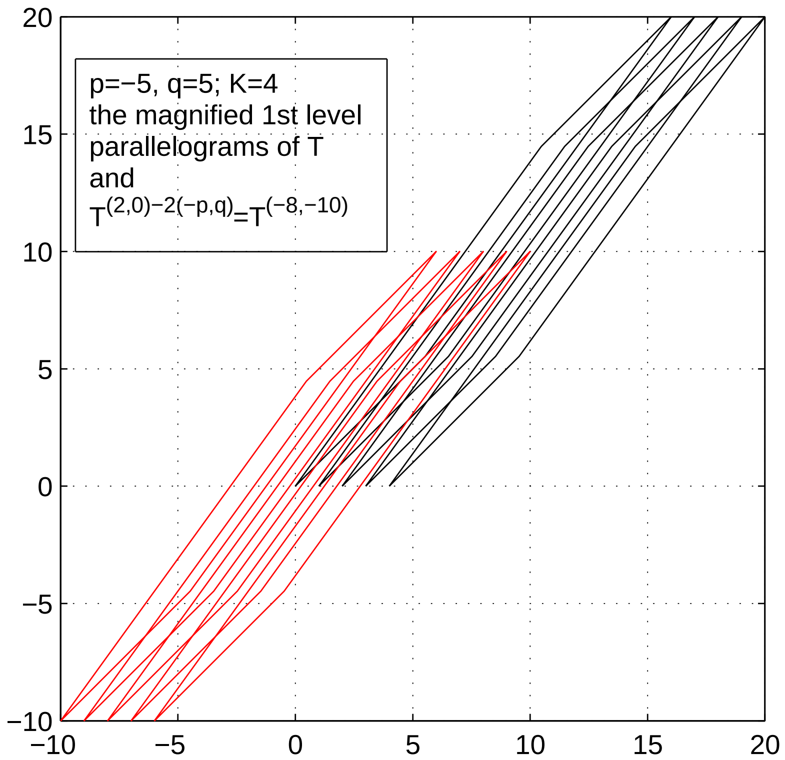

FIGURE 17. The magnified 1st level parallelograms of T and its 2nd layer neighbor $T^{(-8,-10)}$.

Step 7.

Claim. For $k = -1, 0, 1$, the k-th layer neighbors are edge neighbors.

Reason:

- Union of totally disconnected closed sets is totally disconnected $\Rightarrow$ some of the neighbors in the $0, 1, -1$-st layers are edge neighbors.
- Neighbors in the 0-th layer and $1, -1$-st layers are of the same type.

QED

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Proof of Theorem 3.

- Find bounding polygons for the tiles. Through trial and error find P such that $f_i(P) \subset P$.
- To see that there are at most 5 layers of neighbors, at most 2 in each layer, use bounding polygons to separate the non-neighbors.
- Use radix expansion to show that there are exactly 2 neighbors in each of the 5 layers.
- To see that the *2, -2*nd-layers are vertex neighbors, repeatedly magnify the ℓ-level approximations of T and $T^{(2,0)-2(|p|,q)}$. The intersection of the magnified diagrams remains unchanged. So the actual intersection is a point.

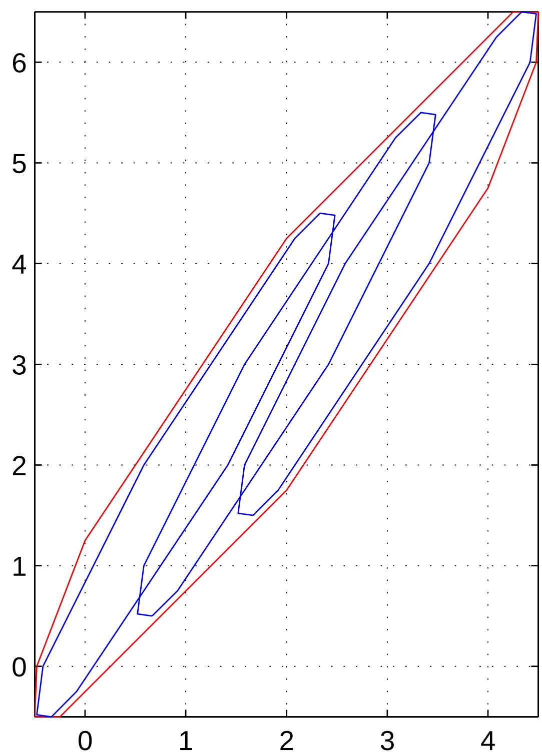

FIGURE 18. A bounding polygon of the exceptional tile.

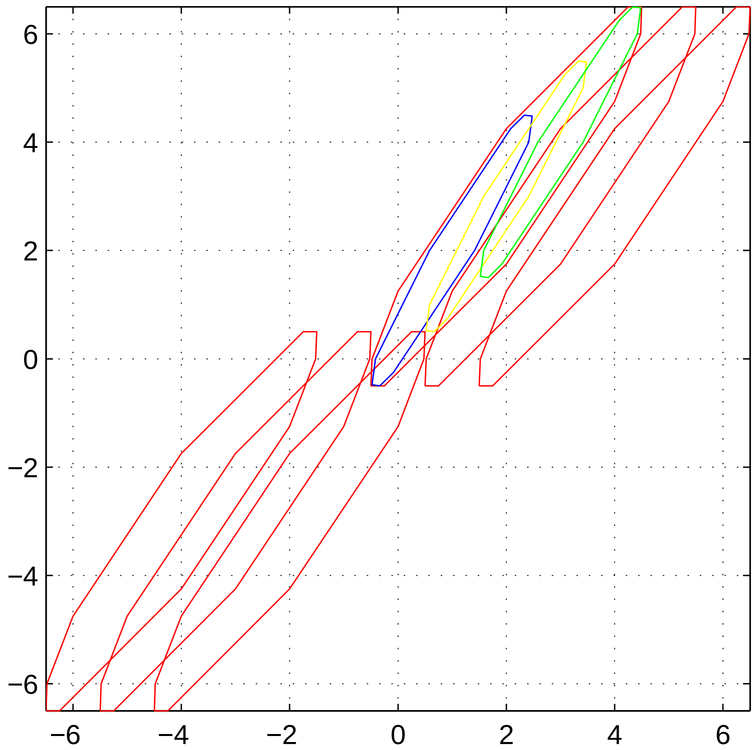

FIGURE 19. 'Vertex neighbor' proof.