Joint work with M. T. Barlow, R. F. Bass, T. Kumagai

UNIQUENESS OF LOCALLY INVARIANT LAPLACIAN, DIRICHLET FORM AND BROWNIAN MOTION ON SIERPINSKI CARPETS

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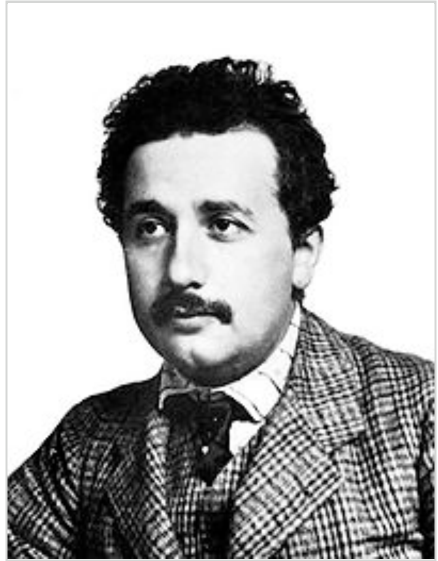
Annus Mirabilis papers

From Wikipedia, the free encyclopedia
(Redirected from Annus Mirabilis Papers)

The **Annus Mirabilis Papers** (from Latin *annus mīrābilis*, 'extraordinary year') are the papers of Albert Einstein published in the "*Annalen der Physik*" scientific journal in 1905. These four articles contributed substantially to the foundation of modern physics and changed views on space, time, and matter. The *Annus Mirabilis* is often called the "Miracle Year" in English or in German, the *"Wunderjahr"*.[1]



Einstein, in 1905, when he wrote the "*Annus Mirabilis Papers*"

Contents

Einstein's Annus Mirabilis 1905 papers:

- Matter and energy equivalence ($\boldsymbol{E = mc^2}$)
- Special relativity (Minkowski 1907)
- Photoelectric effect (Nobel prize in Physics 1921)
- Brownian motion

Brownian motion:
Thiele (1880), Bachelier (1900)
Einstein (1905), Smoluchowski (1906)
Wiener (1920'), Doob, Feller, Levy, Kolmogorov (1930'),
Doeblin, Dynkin, Hunt, Ito ...

Wiener process in $\mathbb{R}^n$ satisfies $\frac{1}{n}\mathbb{E}|W_t|^2 = t$ and has a Gaussian transition density:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

$$distance \sim \sqrt{time}$$

"Einstein space–time relation for Brownian motion"

Gaussian transition density :

$$\boldsymbol{p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)}$$

De Giorgi-Nash-Moser estimates for elliptic and parabolic PDEs;
Li-Yau (1986) type estimates on a geodesically complete Riemannian manifold with Ricci$\geqslant \mathbf{0}$:

$$\boldsymbol{p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)}$$

$$\boldsymbol{distance \sim \sqrt{time}}$$

Gaussian:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

Li-Yau Gaussian-type:

$$p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)$$

Sub-Gaussian:

$$p_t(x,y) \sim \frac{1}{t^{d_f/d_w}} \exp\left(-c\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right)$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

Brownian motion on $\mathbb{R}^d$: $\mathbb{E}|X_t - X_0| = ct^{1/2}$.

Anomalous diffusion: $\mathbb{E}|X_t - X_0| = o(t^{1/2})$, or (in regular enough situations),

$$\mathbb{E}|X_t - X_0| \approx t^{1/d_w}$$

with $d_w > 2$.

Here d_w is the so-called **walk dimension** (should be called "**walk index**" perhaps).

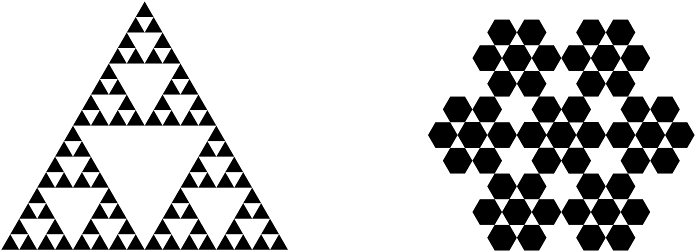
This phenomena was first observed by mathematical physicists working in the transport properties of disordered media, such as (critical) percolation clusters.

$$p_t(x,y) \sim \frac{1}{t^{d_f/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

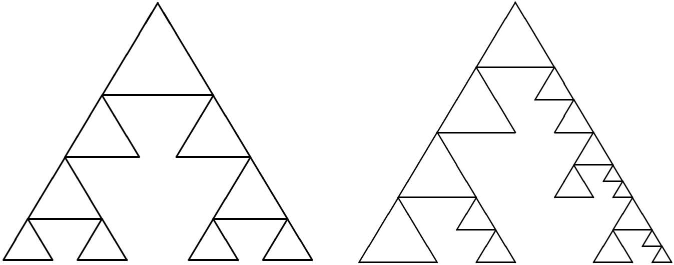
$$d_f = \text{Hausdorff dimension}$$
$$d_w = \text{``walk dimension''}$$
$$\frac{2d_f}{d_w} = \text{``spectral dimension''}$$

First example: **Sierpiński gasket**; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)

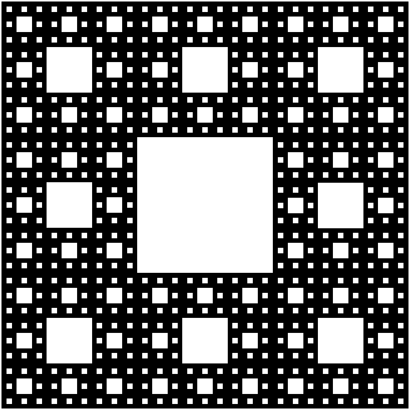


The Sierpinski gasket (left), and a typical nested fractal, the Lindstrøm snowflake (right)

Existence and uniqueness of self-similar diffusions in finitely ramified case can be reduced to a nonlinear eigenvalue problem: Sabot, Lindstrøm, Metz, Peirone.



Simple examples: cut Sierpiński gaskets [Hambly,Metz,T]



Sierpiński carpet

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UNIQUENESS OF BROWNIAN MOTION ON SIERPINSKI CARPETS

M. T. BARLOW[1], R. F. BASS[2], T. KUMAGAI[3], A. TEPLYAEV[4]

Abstract. We prove that, up to scalar multiples, there exists only one local regular Dirichlet form on a generalized Sierpinski carpet that is invariant with respect to the local symmetries of the carpet. Consequently for each such fractal the law of Brownian motion is uniquely determined and the notion of Laplacian is well defined.

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[4]partially supported by NSF grant DMS-0505622.

Contents

1. Introduction

Let $\boldsymbol{F}$ be a GSC and $\boldsymbol{\mu}$ the usual Hausdorff measure on $\boldsymbol{F}$. Let $\mathfrak{E}$ be the set of non-zero local regular conservative Dirichlet forms on $\boldsymbol{L^2(F,\mu)}$ which are **invariant with respect to all the local symmetries** of $\boldsymbol{F}$.

Theorem 1.1. *Let $\boldsymbol{F} \subset \mathbb{R}^{\boldsymbol{d}}$ be a GSC. Then, up to scalar multiples, $\mathfrak{E}$ consists of at most one element. Further, this one element of $\mathfrak{E}$ is self-similar.*

Proposition 1.2. *The Dirichlet forms constructed in [BB89, BB99] and [KZ92] are in $\mathfrak{E}$.*

Corollary 1.3. *The Dirichlet forms constructed in [BB89, BB99] and [KZ92] are (up to a constant) the same, and satisfy scale invariance (i.e. self-similar).*

Corollary 1.4. *If $\boldsymbol{X}$ is a continuous non-degenerate symmetric strong Markov process, whose state space is $\boldsymbol{F}$, and whose Dirichlet form is invariant with respect to the local symmetries of $\boldsymbol{F}$, then the law of $\boldsymbol{X}$, which is a Feller process, is uniquely defined, up to scalar multiples of the time parameter, for each initial point $\boldsymbol{x} \in \boldsymbol{F}$.*

We do not assume the heat kernel exists, or even that the semi-group is Feller, or that the Dirichlet form is irreducible.

The idea of our proof is the following. The main work is showing that if $\mathcal{A}, \mathcal{B}$ are any two Dirichlet forms in $\mathfrak{E}$, then they are comparable. We then let λ be the largest positive real such that $\mathcal{C} = \mathcal{A} - \lambda\mathcal{B} \geq 0$. If $\mathcal{C}$ were also in $\mathfrak{E}$, then $\mathcal{C}$ would be comparable to $\mathcal{B}$, and so there would exist $\varepsilon > 0$ such that $\mathcal{C} - \varepsilon\mathcal{B} \geq 0$, contradicting the definition of λ. In fact we cannot be sure that $\mathcal{C}$ is closed, so instead we consider $\mathcal{C}_\delta = (1+\delta)\mathcal{A} - \lambda\mathcal{B}$, which is easily seen to be in $\mathfrak{E}$. We then need uniform estimates in δ to obtain a contradiction.

A key point here is that the constants in the Harnack inequality, and consequently also the heat kernel bounds, only depend on the GSC F, and not on the particular element of $\mathfrak{E}$. This means that we need to be careful about the dependencies of the constants.

2. Preliminaries

2.1. Some general properties of Dirichlet forms.

Theorem 2.1. *Suppose that* $(\mathcal{A},\mathcal{F})$, $(\mathcal{B},\mathcal{F})$ *are regular* **local** *conservative irreducible Dirichlet forms on* $L^2(F,m)$ *and that*

$$\mathcal{A}(u,u) \le \mathcal{B}(u,u) \quad \text{for all } u \in \mathcal{F}.$$

Let $\delta > 0$, *and* $\mathcal{E} = (1+\delta)\mathcal{B}-\mathcal{A}$. *Then* $(\mathcal{E},\mathcal{F})$ *is a regular local conservative irreducible Dirichlet form on* $L^2(F,m)$.

Since $\mathcal{E}$ is local regular, $\mathcal{E}(f,f)$ can be written in terms of a measure $\Gamma(f,f)$, the energy measure of f, as follows. Let $\mathcal{F}_b$ be the elements of $\mathcal{F}$ that are essentially bounded. If $f \in \mathcal{F}_b$, $\Gamma(f,f)$ is the unique smooth Borel measure on F satisfying $\int_F g\,d\Gamma(f,f) = 2\mathcal{E}(f,fg) - \mathcal{E}(f^2,g)$, $g \in \mathcal{F}_b$.

Lemma 2.2. *If* $\mathcal{E}$ *is a local regular Dirichlet form with domain* $\mathcal{F}$, *then for any* $f \in \mathcal{F} \cap L^\infty(F)$ *we have* $\Gamma(f,f)(A) = 0$, *if* $A = \{x \in F : f(x)=0\}$.

We call a function $u : \mathbb{R}_+ \times F \longrightarrow \mathbb{R}$ **caloric** in D in probabilistic sense if $u(t,x) = \mathbb{E}^x[f(X_{t\wedge\tau_D})]$ for some bounded Borel $f : F \longrightarrow \mathbb{R}$, which is the **solution to the heat equation** with boundary data defined by $f(x)$ outside of D and the initial data defined by $f(x)$ inside of D. Let $\overline{T}_t$ be the semigroup of X killed on exiting D, which can be either defined probabilistically as above or, equivalently, in the Dirichlet form sense according to Theorems 4.4.3 and A.2.10 in [FOT].

Proposition 2.3. *Let* $(\mathcal{E},\mathcal{F})$ *and* D *satisfy the above conditions, and let* $f \in \mathcal{F}$ *be bounded and* $t \geq 0$*. Then*

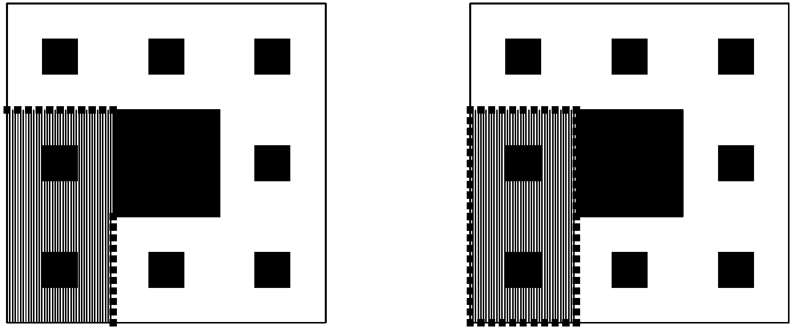
$$\mathbb{E}^x[f(X_{t\wedge\tau_D})] = h(x) + \overline{T}_t f_D$$

q.e., where $h(x) = \mathbb{E}^x[f(X_{\tau_D})]$ *is the harmonic function that consides with* f *on* D^c*, and* $f_D(x) = f(x) - h(x)$*.*

2.2. **Generalized Sierpinski carpets.** Let $d \geq 2$, $F_0 = [0,1]^d$, and let $L_F \in \mathbb{N}$, $L_F \geq 3$, be fixed. For $n \in \mathbb{Z}$ let $\mathcal{Q}_n$ be the collection of closed cubes of side L_F^{-n} with vertices in $L_F^{-n}\mathbb{Z}^d$. For $A \subseteq \mathbb{R}^d$, set $\mathcal{Q}_n(A) = \{Q \in \mathcal{Q}_n : \mathrm{int}(Q) \cap A \neq \emptyset\}.$ Let Ψ_Q be the orientation preserving affine map from F_0 onto Q. Let $1 \leq m_F \leq L_F^d$ be an integer, and let F_1 be the union of m_F distinct elements of $\mathcal{Q}_1(F_0)$.

- (H1) (Symmetry) F_1 is preserved by the isometries of the unit cube F_0.
- (H2) (Connectedness) $\mathrm{Int}(F_1)$ is connected.
- (H3) (Non-diagonality) Let $m \geq 1$ and $B \subset F_0$ be a cube of side length $2L_F^{-m}$, which is the union of 2^d distinct elements of $\mathcal{Q}_m$. Then if $\mathrm{int}(F_1 \cap B)$ is non-empty, it is connected.
- (H4) (Borders included) F_1 contains the line segment $\{x : 0 \leq x_1 \leq 1, x_2 = ... = x_d = 0\}$.

Given $S \in \mathcal{S}_n$, $f : S \to \mathbb{R}$ and $g : F \to \mathbb{R}$ we define the unfolding and restriction operators by $U_S f = f \circ \varphi_S, \quad R_S g = g|_S$, where $\varphi_S : F \to S$.



Definition 2.4. We define the *length* and *mass* scale factors of F to be L_F and m_F respectively. The Hausdorff dimension of F is $d_f = d_f(F) = \log m_F / \log L_F$.

Let D_n be the network of diagonal crosswires obtained by joining each vertex of a cube $Q \in \mathcal{Q}_n$ to a vertex at the center of the cube by a wire of unit resistance. Write R_n^D for the resistance across two opposite faces of D_n. There exists ρ_F and C_i, depending only on the dimension d, such that $\rho_F \leq L_F^2 / m_F$ and

$$C_1 \rho_F^n \leq R_n^D \leq C_2 \rho_F^n.$$

2.3. **F-invariant Dirichlet forms.** Let $(\mathcal{E},\mathcal{F})$ be a regular local Dirichlet form on $L^2(F,\mu)$. Let $S\in\mathcal{S}_n$. We set

$$\mathcal{E}^S(g,g)=\frac{1}{m_F^n}\mathcal{E}(U_Sg,U_Sg).$$

and define the domain of $\mathcal{E}^S$ to be $\mathcal{F}^S=\{g: g \text{ maps } S \text{ to } \mathbb{R}, U_Sg\in\mathcal{F}\}$.

Definition 2.5. Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on $L^2(F,\mu)$. We say that $\mathcal{E}$ *is invariant with respect to all the local symmetries of* F (F-invariant or $\mathcal{E}\in\mathfrak{E}$) if

- (1) If $S\in\mathcal{S}_n(F)$, then $U_SR_Sf\in\mathcal{F}$ for any $f\in\mathcal{F}$.
- (2) Let $n\geq 0$ and S_1,S_2 be any two elements of $\mathcal{S}_n$, and let Φ be any isometry of $\mathbb{R}^d$ which maps S_1 onto S_2. If $f\in\mathcal{F}^{S_2}$, then $f\circ\Phi\in\mathcal{F}^{S_1}$ and $\mathcal{E}^{S_1}(f\circ\Phi,f\circ\Phi)=\mathcal{E}^{S_2}(f,f)$.
- (3) $\mathcal{E}(f,f)=\sum_{S\in\mathcal{S}_n(F)}\mathcal{E}^S(R_Sf,R_Sf)$ for all $f\in\mathcal{F}$

Lemma 2.6. *Let* $(\mathcal{A},\mathcal{F}_1)$, $(\mathcal{B},\mathcal{F}_2)\in\mathfrak{E}$ *with* $\mathcal{F}_1=\mathcal{F}_2$ *and* $\mathcal{A}\geq\mathcal{B}$. *Then* $\mathcal{C}=(1+\delta)\mathcal{A}-\mathcal{B}\in\mathfrak{E}$ *for any* $\delta>0$.

$$\Theta f = \frac{1}{m_F^n} \sum_{S \in \mathcal{S}_n(F)} U_S R_S f.$$

Note that Θ is a projection operator because $\Theta^2 = \Theta$. It is bounded on $C(F)$ and is an orthogonal projection on $L^2(F, \mu)$.

Proposition 2.7. *Assume that $\mathcal{E}$ is a local regular Dirichlet form on F, T_t is its semigroup, and $U_S R_S f \in \mathcal{F}$ whenever $S \in \mathcal{S}_n(F)$ and $f \in \mathcal{F}$. Then the following, for all $f, g \in \mathcal{F}$, are equivalent:*

(a): $\mathcal{E}(f, f) = \sum_{S \in \mathcal{S}_n(F)} \mathcal{E}^S(R_S f, R_S f)$

(b): $\mathcal{E}(\Theta f, g) = \mathcal{E}(f, \Theta g)$

(c): $T_t \Theta f = \Theta T_t f$

3. The Barlow-Bass and Kusuoka-Zhou Dirichlet forms

Theorem 3.1. *Each $\boldsymbol{\mathcal{E}_{BB}}$ and $\boldsymbol{\mathcal{E}_{KZ}}$ is in $\mathfrak{E}$.*

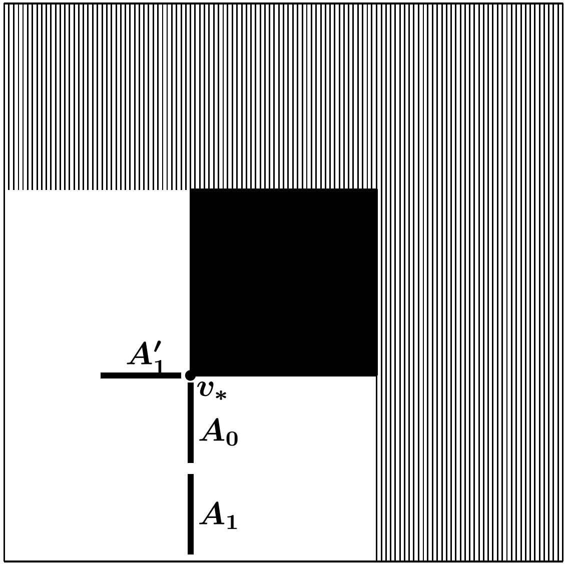
4. Diffusions associated with $\boldsymbol{F}$-invariant Dirichlet forms

Let $\boldsymbol{X} = \boldsymbol{X}^{(\boldsymbol{\mathcal{E}})}$ be an $\boldsymbol{\mathcal{E}}$-diffusion, $\boldsymbol{T_t} = \boldsymbol{T_t^{(\mathcal{E})}}$ be the semigroup of $\boldsymbol{X}$ and $\mathbb{P}^{\boldsymbol{x}} = \mathbb{P}^{\boldsymbol{x},(\boldsymbol{\mathcal{E}})}$, $\boldsymbol{x} \in \boldsymbol{F} - \boldsymbol{\mathcal{N}_0}$, the associated probability laws. Here $\boldsymbol{\mathcal{N}_0}$ is a properly exceptional set for $\boldsymbol{X}$. Ultimately we will be able to define $\mathbb{P}^{\boldsymbol{x}}$ for all $\boldsymbol{x} \in \boldsymbol{F}$, so that $\boldsymbol{\mathcal{N}_0} = \emptyset$.

4.1. Reflected processes and Markov property.

Theorem 4.1. *Let $\boldsymbol{S} \in \boldsymbol{\mathcal{S}_n}(\boldsymbol{F})$. Let $\boldsymbol{Z} = \boldsymbol{\varphi_S}(\boldsymbol{X})$. Then $\boldsymbol{Z}$ is a $\boldsymbol{\mu_S}$-symmetric Markov process with Dirichlet form $(\boldsymbol{\mathcal{E}^S}, \boldsymbol{\mathcal{F}^S})$, and semigroup $\boldsymbol{T_t^Z f} = \boldsymbol{R_S T_t U_S f}$. Write $\widetilde{\mathbb{P}}^{\boldsymbol{y}}$ for the laws of $\boldsymbol{Z}$; these are defined for $\boldsymbol{y} \in \boldsymbol{S} - \boldsymbol{\mathcal{N}_2^Z}$, where $\boldsymbol{\mathcal{N}_2^Z}$ is a properly exceptional set for $\boldsymbol{Z}$. There exists a properly exceptional set $\boldsymbol{\mathcal{N}_2}$ for $\boldsymbol{X}$ such that for any Borel set $\boldsymbol{A} \subset \boldsymbol{F}$,*

$$\widetilde{\mathbb{P}}^{\boldsymbol{\varphi_S(x)}}(\boldsymbol{Z_t} \in \boldsymbol{A}) = \mathbb{P}^{\boldsymbol{x}}(\boldsymbol{X_t} \in \boldsymbol{\varphi_S^{-1}}(\boldsymbol{A})), \quad \boldsymbol{x} \in \boldsymbol{F} - \boldsymbol{\mathcal{N}_2}.$$



The half-face A_1 corresponds to a “slide move”,
and the half-face A_1' corresponds to a “corner move”,
analogues of the “corner” and “knight’s” moves in [BB89].

4.2. **Moves by Z and X.** The key idea, as in [BB99], is to prove that certain 'moves' of the process in F have probabilities which can be bounded below by constants depending only on the dimension d. We begin by looking at the process $Z = \varphi_S(X)$ for some $S \in \mathcal{S}_n$, where $n \geq 0$.

Let $1 \leq i, j \leq d$, with $i \neq j$, assume $n = 0$ and $S = F$, and

$H_i(t) = \{x = (x_1, \dots, x_d) : x_i = t\}$, $t \in \mathbb{R}$;

$L_i = H_i(0) \cap [0, 1/2]^d$;

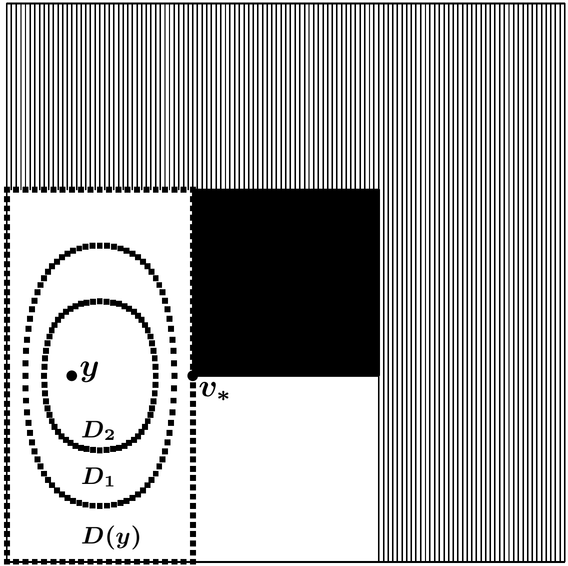
$M_{ij} = \{x \in [0,1]^d : \ x_i = 0, \ \frac{1}{2} \leq x_j \leq 1, \ \text{and } 0 \leq x_k \leq \frac{1}{2} \text{ for } k \neq j\}$.

$$\partial_e S = S \cap (\cup_{i=1}^d H_i(1)), \qquad D = S - \partial_e S.$$

Proposition 4.2. *There exists a constant q_0, depending only on the dimension d, such that for any $n \geq 0$*

$$\widetilde{\mathbb{P}}^x(T^Z_{L_j} < \tau^Z_D) \geq q_0, \quad x \in L_i \cap E_D,$$

$$\widetilde{\mathbb{P}}^x(T^Z_{M_{ij}} < \tau^Z_D) \geq q_0, \quad x \in L_i \cap E_D.$$



y
v_*
D_2
D_1
$D(y)$

Define

$$\mathcal{F}_{D_1} = \{f \in \mathcal{F} : \mathrm{supp}(f) \subset D_1\}$$

and denote by $\mathcal{E}_{D_1}$ the associated Dirichlet form and by $T_t^{D_1}$ the associated semigroup, which are the Dirichlet form and the semigroup of the process X killed on exiting D_1, according to Theorems 4.4.3 and A.2.10 in [FOT].

Lemma 4.3. *Let D_1, D_2 be as above.*
(a) Let $f \in \mathcal{F}_{D_1}$. Then $\Theta^{D_1} f \in \mathcal{F}_{D_1}$. Moreover, for all $f, g \in \mathcal{F}_{D_1}$ we have

$$\mathcal{E}_{D_1}(\Theta^{D_1} f, g) = \mathcal{E}_{D_1}(f, \Theta^{D_1} g)$$

and $T_t^{D_1} \Theta^{D_1} f = \Theta^{D_1} T_t^{D_1} f$.
(b) If $h \in \mathcal{F}_{D_1}$ is harmonic (in the Dirichlet form sense) in D_2 then $\Theta^{D_1} h$ is harmonic (in the Dirichlet form sense) in D_2.
(c) If u is caloric in D_2, in the sense of Proposition 2.3, then $\Theta^{D_1} u$ is also caloric in D_2.

Lemma 4.4. *We have for any bounded Borel function $f : D_1 \to \mathbb{R}$ and all $0 \le t \le \infty$ that $\mathbb{E}^y f(X_{t\wedge V} | \mathcal{F}^Z_{t\wedge V}) = \left(\Theta^{D_1} f\right)(Z_{t\wedge V})$.*

Thus the conditional distribution of X_t given $\mathcal{F}^Z_{t\wedge\tau}$ is $\sum_{i=1}^k p_i(t)\delta_{W_i(t\wedge\tau)}$.

To describe the intuitive picture, we call the W_i "particles." If Z_t is in a lower dimensional face, then there can be fewer than m distinct points $W_i(t)$, because some of them coincide and we can have $X_t = W_i(t) = W_j(t)$ for $i \neq j$. We call such a situation a "collision."

Lemma 4.5. *The processes $p_i(t)$ satisfy the following:*
(a) If T is any $(\mathcal{F}^Z_t)$ stopping time satisfying $T \le \tau$ on $\{T < \infty\}$ then there exists $\delta(\omega) > 0$ such that

$$p_i(T+h) = p_i(T) \quad \text{for } 0 \le h < \delta.$$

(b) Let T be any $(\mathcal{F}^Z_t)$ stopping time satisfying $T \le \tau$ on $\{T < \infty\}$. Then for each $i = 1, \ldots k$,

$$p_i(T) = \lim_{s\to T-} M_i(T)^{-1} \sum_{j\in J_i(T)} p_j(s),$$

where $M_i(t) = N_n(W_i(t))$ for $0 \le t < \tau$, which is the number of elements of $\mathcal{S}_n$ that contain $W_i(t)$.

Proposition 4.6. *There exists a constant $q_1 > 0$, depending only on d, such that if $x \in A_0 \cap E_D$ and $T_0 \leq \tau$ is a finite $(\mathcal{F}_t^Z)$ stopping time, then*

$$\mathbb{P}^x(X_{T_0} \in S | \mathcal{F}_{T_0}^Z) \geq q_1.$$

Hence

$$\mathbb{P}^x(T_{A_1}^X \leq \tau) \geq q_0 q_1.$$

4.3. Properties of X.

Lemma 4.7. *Let $U \subset F$ be open and non-empty. Then $\mathbb{P}^x(T_U < \infty) = 1$, q.e.*

4.4. Coupling.

We say that $x, y \in F$ are m-*associated* if there exists an isometry of a cube in $\mathcal{Q}_m$ containing x onto a cube in $\mathcal{Q}_m$ containing y that maps x onto y; we write $x \sim_m y$. It is easy to see that $x \sim_m y$ if and only if $\varphi_S(x) = \varphi_S(y)$ for each $S \in \mathcal{S}_m$.

The coupling result we want is:

Proposition 4.8. *Let $x_1, x_2 \in F$ with $x_1 \sim_n x_2$, where $x_1 \in S_1 \in \mathcal{S}_n(F)$, $x_2 \in S_2 \in \mathcal{S}_n(F)$, and let $\Phi = \varphi_{S_1}|_{S_2}$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying processes X_k, $k = 1, 2$ and Z with the following properties.*
(a) Each X_k is an $\mathcal{E}$-diffusion started at x_k.
(b) $Z = \varphi_{S_2}(X_2) = \Phi \circ \varphi_{S_1}(X_1)$.
(c) X_1 and X_2 are conditionally independent given Z.

Given a pair of $\mathcal{E}$-diffusions $X_1(t)$ and $X_2(t)$ we define the coupling time

$$T_C(X_1, X_2) = \inf\{t \geq 0 : X_1(t) = X_2(t)\}.$$

Given Propositions 4.6 and 4.8 we can now use the same arguments as in [BB99] to couple copies of X started at points $x, y \in F$, provided that $x \sim_m y$ for some $m \geq 1$.

Theorem 4.9. *Let $r > 0$, $\varepsilon > 0$ and $r' = r/L_F^2$. There exist constants q_3 and δ, depending only on the GSC F, such that the following hold:*
(a) Suppose x_1, $x_2 \in F$ with $||x_1 - x_2||_\infty < r'$ and $x_1 \sim_m x_2$ for some $m \geq 1$. There exist $\mathcal{E}$-diffusions $X_i(t)$, $i = 1, 2$, with $X_i(0) = x_i$, such that, writing

$$\tau_i = \inf\{t \geq 0 : X_i(t) \notin B(x_1, r)\},$$

we have

$$\mathbb{P}\big(T_C(X_1, X_2) < \tau_1 \wedge \tau_2\big) > q_3.$$

(b) If in addition $||x_1 - x_2||_\infty < \delta r$ and $x_1 \sim_m x_2$ for some $m \geq 1$ then

$$\mathbb{P}\big(T_C(X_1, X_2) < \tau_1 \wedge \tau_2\big) > 1 - \varepsilon.$$

4.5. **Elliptic Harnack inequality.** X satisfies the *elliptic Harnack inequality* if there exists a constant c_1 such that the following holds: for any ball $B(x,R)$, whenever u is a non-negative harmonic function on $B(x,R)$ then there is a quasi-continuous modification $\tilde{u}$ of u that satisfies

$$\sup_{B(x,R/2)} \tilde{u} \leq c_1 \inf_{B(x,R/2)} \tilde{u}.$$

Lemma 4.10. *Let $r \in (0,1)$, and h be bounded and harmonic in $B = B(x_0, r)$. Then there exists $\theta > 0$ such that*

$$|h(x) - h(y)| \leq C\Big(\frac{|x-y|}{r}\Big)^{\theta}(\sup_B |h|), \quad x, y \in B(x_0, r/2), \quad x \sim_m y.$$

Proposition 4.11. *There exists a set $\mathcal{N}$ of $\mathcal{E}$-capacity 0 such that the Lemma above holds for all $x, y \in B(x_0, r/2) - \mathcal{N}$.*

Proposition 4.12. *EHI holds for $\mathcal{E}$, with constants depending only on F.*

Corollary 4.13. *If $\mathcal{E} \in \mathfrak{E}$ then*
(a) $\mathcal{E}$ is irreducible;
(b) if $\mathcal{E}(f,f) = 0$ then f is a.e. constant;
(c) $||\mathcal{E}|| > 0$, where $||\mathcal{E}||$ is the effective resistance between two opposite faces of the GSC.

4.6. **Resistance estimates.** Let now $\mathcal{E} \in \mathfrak{E}_1$. Let $S \in \mathcal{S}_n$ and let $\gamma_n = \gamma_n(\mathcal{E})$ be the conductance across S. That is, if $S = Q \cap F$ for $Q \in \mathcal{Q}_n(F)$ and $Q = \{a_i \le x_i \le b_i, i = 1, \dots, d\}$, then

$$\gamma_n = \inf\{\mathcal{E}^S(u,u) : u \in \mathcal{F}^S, u\,|_{\{x_1=a_1\}} = 0, u\,|_{\{x_1=b_1\}} = 1\}.$$

Note that γ_n does not depend on S, and that $\gamma_0 = 1$. Write $v_n = v_n^{\mathcal{E}}$ for the minimizing function. We remark that from the results in [BB3, McG] we have

$$C_1 \rho_F^n \le \gamma_n(\mathcal{E}_{BB}) \le C_2 \rho_F^n.$$

Proposition 4.14. *Let $\mathcal{E} \in \mathfrak{E}_1$. Then for $n, m \ge 0$*

$$\gamma_{n+m}(\mathcal{E}) \ge C_1 \gamma_m(\mathcal{E}) \rho_F^n.$$

We define a 'time scale function' H for $\mathcal{E}$ ...

We say $\mathcal{E}$ satisfies the condition RES(H, c_1, c_2) if for all x, $r \in (0, L_F^{-1})$,

$$c_1 \frac{H(r)}{r^\alpha} \le R_{\text{eff}}(B(x_0,r), B(x_0,2r)^c) \le c_2 \frac{H(r)}{r^\alpha}. \quad [RES(H,c_1,c_2)]$$

Proposition 4.15. *There exist constants C_1, C_2, depending only on F, such that $\mathcal{E}$ satisfies RES(H, C_1, C_2).*

4.7. **Exit times, heat kernel and energy estimates.** We write h for the inverse of H, and $V(x,r) = \mu(B(x,r))$. We say that $p_t(x,y)$ satisfies HK$(H;\eta_1,\eta_2,c_0)$ if for $x,y \in F$, $0 < t \le 1$,

$$p_t(x,y) \ge c_0^{-1} V(x,h(t))^{-1} \exp(-c_0 (H(d(x,y))/t)^{\eta_1}),$$
$$p_t(x,y) \le c_0 V(x,h(t))^{-1} \exp(-c_0^{-1} (H(d(x,y))/t)^{\eta_2}).$$

Theorem 4.16 (GT,BBKT). *Let $H : [0,2] \to [0,\infty)$ be a strictly increasing function with $H(1) \in (0,\infty)$ that satisfies ... Then TFAE:*
(a) $(\mathcal{E},\mathcal{F})$ satisfies (VD), (EHI) and $(RES(H,c_1,c_2))$
(b) $(\mathcal{E},\mathcal{F})$ satisfies $(HK(\alpha,H;\eta_1,\eta_2,c_0))$
Further the constants in each implication are effective.

By saying that the constants are 'effective' we mean that if, for example (a) holds, then the constants η_i, c_0 in (b) depend only on the constants c_i in (a), and the constants in (VD), (EHI) and ...

Theorem 4.17.
X has a transition density $p_t(x,y)$ which satisfies HK$(H;\eta_1,\eta_2,C)$, with $\eta_1 = 1/(\beta_0 - 1)$, $\eta_2 = 1/(\beta' - 1)$ with the constants depending only on F.

Let

$$\begin{aligned} J_r(f) &= r^{-\alpha} \int_F \int_{B(x,r)} |f(x) - f(y)|^2 d\mu(x) d\mu(y), \\ N_H^r(f) &= H(r)^{-1} J_r(f), \\ N_H(f) &= \sup_{0<r\le 1} N_H^r(f), \\ W_H &= \{u \in L^2(F,\mu) : N_H(f) < \infty\}. \end{aligned}$$

Theorem 4.18 (KS,BBKT). *Let* H *satisfy ... Suppose* p_t *satisfies* *HK*(H, η_1, η_2, C_0). *Then*

$$C_1 \mathcal{E}(f,f) \le \limsup_{j\to\infty} N_H^{r_j}(f) \le N_H(f) \le C_2 \mathcal{E}(f,f) \quad \textit{for all } f \in W_H,$$

where the constants C_i *depend only on the constants in* $HK(H; \eta_1, \eta_2, C)$, *and in ... Further,*

$$\mathcal{F} = W_H.$$

Theorem 4.19. *Let* $(\mathcal{E}, \mathcal{F}) \in \mathfrak{E}_1$.
(a) There exist constants $C_1, C_2 > 0$ *such that for all* $r \in [0, 1]$,

$$C_1 H_0(r) \le H(r) \le C_2 H_0(r), \qquad H_0(r) = r^{\beta_0}.$$

(b) $W_H = W_{H_0}$, *and there exist constants* C_3, C_4 *such that*

$$C_3 N_{H_0}(f) \le \mathcal{E}(f, f) \le C_4 N_{H_0}(f) \quad \text{for all } f \in W_H.$$

(c) $\mathcal{F} = W_{H_0} = \mathcal{F}_0$.

Remark 4.20. $p_t(x, y)$ satisfies $\mathrm{HK}(H_0, \eta_1, \eta_1, C)$ with $\eta_1 = 1/(\beta_0 - 1)$.

5. Uniqueness

Definition 5.1. Let $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}} \in \mathfrak{E}$. We say $\boldsymbol{\mathcal{A}} \leq \boldsymbol{\mathcal{B}}$ if

$$\boldsymbol{\mathcal{B}}(\boldsymbol{u}, \boldsymbol{u}) - \boldsymbol{\mathcal{A}}(\boldsymbol{u}, \boldsymbol{u}) \geq \mathbf{0} \text{ for all } \boldsymbol{u} \in \boldsymbol{W}.$$

For $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}} \in \mathfrak{E}$ define

$$\sup(\boldsymbol{\mathcal{B}}|\boldsymbol{\mathcal{A}}) = \sup\left\{\frac{\boldsymbol{\mathcal{B}}(\boldsymbol{f}, \boldsymbol{f})}{\boldsymbol{\mathcal{A}}(\boldsymbol{f}, \boldsymbol{f})} : \boldsymbol{f} \in \boldsymbol{W}\right\},$$

$$\boldsymbol{h}(\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}) = \log\left(\frac{\sup(\boldsymbol{\mathcal{B}}|\boldsymbol{\mathcal{A}})}{\sup(\boldsymbol{\mathcal{A}}|\boldsymbol{\mathcal{B}})}\right).$$

Note that $\boldsymbol{h}$ is Hilbert's projective metric and we have $\boldsymbol{h}(\boldsymbol{\theta}\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}) = \boldsymbol{h}(\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}})$ for any $\boldsymbol{\theta} \in (\mathbf{0}, \infty)$, and $\boldsymbol{h}(\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}) = \mathbf{0}$ if and only if $\boldsymbol{\mathcal{A}}$ is a nonzero constant multiple of $\boldsymbol{\mathcal{B}}$.

Theorem 5.2. *There exists a constant $\boldsymbol{C_F}$, depending only on the GSC $\boldsymbol{F}$, such that if $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}} \in \mathfrak{E}$ then*

$$\boldsymbol{h}(\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}) \leq \boldsymbol{C_F}.$$

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