Curvature densities of self-similar sets

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1. Lipschitz-Killing curvatures of sets with positive reach

Differential geometry:

integrals of k th mean curvatures of a d-dimensional submanifold $M_d \subset \mathbb{R}^d$ with smooth boundary:

$$C_k(M_d) := \int_{\partial M_d} S_{d-1-k}(\varkappa_1, \ldots, \varkappa_{d-1})\, d\mathcal{H}^{d-1}$$

k th Lipschitz-Killing curvature, $k = 0, \ldots, d-1$, where

$$S_l((\varkappa_1, \ldots, \varkappa_{d-1}) := \mathrm{const}(d,l) \sum_{1 \le i_1 \ldots \le i_l \le d-1} \varkappa_1 \ldots \varkappa_l$$

l th symmetric function of principal curvatures $\varkappa_1, \ldots, \varkappa_{d-1}$

Special cases: $k = 0$ total Gauss curvature = Euler characteristic, $k = d-2$ total mean curvature, $k = d-1$ surface area, define additionally for $k = d$: $C_d(M_d) := \mathcal{L}^d(M_d)$ volume

Convex geometry:

$V^k(K)$ k th intrinsic volume of a convex body K; for smooth boundary

$$V^k(K) = C_k(\partial K)$$

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For $\varepsilon > 0$ and $A \subset \mathbb{R}^d$ denote

$$A_\varepsilon := \{x \in \mathbb{R}^d : d(x, A) \le \varepsilon\} .$$

Theorem (Fu 1985)

For any compact $K \subset \mathbb{R}^d$ with $d \le 3$, Lebesgue-a.e. $\varepsilon > 0$ is a regular value of the distance function of K and, hence, the closure of the complement of the the parallel set K_ε has positive reach.

For arbitrary d and compact K with this property define the k th Lipschitz-Killing curvature of the parallel sets K_ε for those ε by

$$C_k(K_\varepsilon) := (-1)^{d-k} C_k\left(\overline{(K_\varepsilon)^c}\right)$$

(consistent definition).

For classical sets K as above we have

$$\lim_{\varepsilon \longrightarrow 0} C_k(K_\varepsilon) = C_k(K) ,$$

for fractal sets explosion. Therefore rescaling is necessary:

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F self-similar set in $\mathbb{R}^d$ with Hausdorff dimension D satisfying OSC

Winter [Thesis 06, printed version 08]: Under the additional assumption of polyconvex neighborhoods F_ε the following limits exist:

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in the "non-arithmetic case" and generally,

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Measure version (Winter):

$$C_k(F,\cdot) := (w)\lim_{\delta\to 0}\frac{1}{|\ln\delta|}\int_\delta^1 \varepsilon^{D-k} C_k(F_\varepsilon,\cdot)\,\frac{1}{\varepsilon}\,d\varepsilon \tag{1}$$

$$= C_k(F)\mathcal{H}^D(F)^{-1}\mathcal{H}^D(F\cap(\cdot))\,. \tag{2}$$

Extensions

Z.: non-polyconvex neighborhoods, random self-similar sets (results for total curvatures)

Winter/Z. (in preparation): measure version in the deterministic non-polyconvex case

Interpretation of $\mathcal{H}^D(F)^{-1}C_k(F)$: some fractal analogue of the pointwise mean curvatures on smooth submanifolds

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(allows to consider more general types of (random) fractals, applications to random fractal tessellations)

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3. Average curvature densities

Let O be from (OSC), $SO := \bigcup_{i=1}^{N} S_i O$ for the generating similarities $S_1, \ldots, S_N$ with contraction rations $r_1, \ldots, r_N$.

For $0 < \varepsilon < \varepsilon_0$ and $x \in F$ let $A_F(x, \varepsilon)$ be a family of sets of such that $A_F(x, \varepsilon) \subset F_\varepsilon \cap B(x, a\varepsilon)$ for some $a > 1$ and

$$S_i(A_F(x, \varepsilon)) = A_{S_i(F)}(S_i(x), r_i \varepsilon) \ , \ i = 1, \ldots, N \, .$$

Examples:

1. $A_F(x, \varepsilon) = F_\varepsilon \cap B(x, a\varepsilon)$
2. $A_F(x, \varepsilon) = F_\varepsilon \cap \Pi_F^{-1}\big(B(x, \varepsilon)\big)$,
   the set of those points from F_ε which have a foot point on F within the ball $B(x, \varepsilon)$
3. $A_F(x, \varepsilon) = \{y \in F_\varepsilon : |y - x| < \varrho_F(y, \varepsilon)\}$,
   where $\varrho_F(y, \varepsilon)$ is determined by $\mathcal{H}^D\big(F \cap B(y, \varrho_F(y, \varepsilon))\big) = \varepsilon^D$

Main result:

For $\mathcal{H}^D$-a.a. $x \in F$ the following limit exists

$$\lim_{\delta\to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{-k} C_k\big(F_\varepsilon, A_F(x,\varepsilon)\big)\, \frac{1}{\varepsilon}\, d\varepsilon$$

and equals the constant

$$\mathcal{H}^D(F)^{-1}\Big(\sum_{i=1}^N r_i^D \ln r_i\Big)^{-1} \int_F \int_{\frac{d(y,(SO)^c)}{2a}}^{\frac{d(y,O^c)}{2a}} \varepsilon^{-k} C_k\big(F_\varepsilon, A_F(y,\varepsilon)\big)\, \frac{1}{\varepsilon}\, d\varepsilon\, \mathcal{H}^D(dy)$$

provided the last integral converges.
The limit agrees with the former local variant $\mathcal{H}^D(F)^{-1} C_k(F)$ if the total fractal curvature $C_k(F)$ exists and the sets $A_F(x,\varepsilon)$ are chosen as in Example 3. $(k = 0, \ldots, d.)$

Heuristic interpretation:
Example 3 corresponds to D-dimensional Minkowski content and Example 2 to D-dimensional Hausdorff measure on F (for $k = d$ this is exact).

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