The duality in the affine action on trees

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Groups acting on trees

Groups acting on trees:

- Action of algebraic groups over local fields on their Bruhat-Tits trees
- Action on rooted trees given by Mealy automata (Burnside groups, groups of intermediate growth, ...)
- Groups acting on the product of two trees (non-residually finite CAT(0) groups, finitely presented torsion-free simple groups, ...)

Automata-transducers

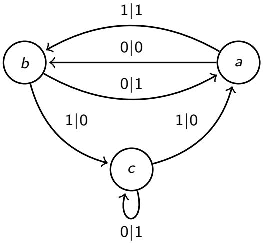
We consider **complete deterministic** Mealy automata.

A finite **automaton** is a quadruple $A = (S, X, \tau, \pi)$, where

- X is a finite set (the input and output alphabet),
- S is a finite set (the set of states),
- $\tau : S \times X \to S$ is the transition map,
- $\pi : S \times X \to X$ is the output map.

An automaton can be identified with a **finite directed labeled graph** with vertices S and arrows

$$s \xrightarrow{x|y} t \text{ iff } \pi(s, x) = y \text{ and } \tau(s, x) = t.$$



Automaton action on a tree

Let X^* be the space of finite words over X. For every state $s \in S$, the automaton A initialized at s produces the transformation $A_s : X^* \to X^*$:

$$A_s(x_1 x_2 \dots x_n) = y_1 y_2 \dots y_n$$

for a path $\textcircled{s} \xrightarrow{x_1|y_1} \bigcirc \xrightarrow{x_2|y_2} \bigcirc \dots \bigcirc \xrightarrow{x_n|y_n} \bigcirc$.

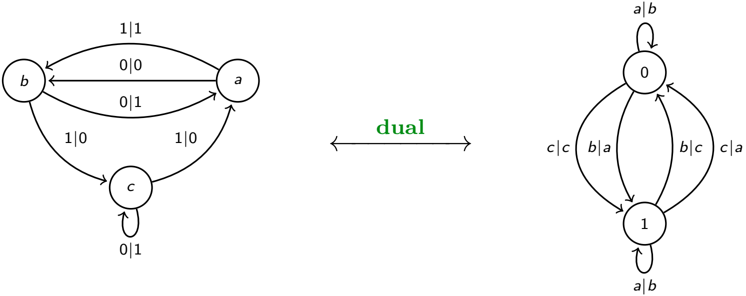
If all A_s are invertible, they generate an **automaton group** G_A. A given group is an automaton group iff there is an automaton structure on the group.

The space X^* has a natural tree structure. Every automaton group acts by automorphisms on this tree.

Dual automata

$(S, X, \tau, \pi) \rightsquigarrow$ automaton A over X with states S
$(X, S, \tau, \pi) \rightsquigarrow$ **dual automaton** ∂A over S with states X

$$s \xrightarrow{x|y} t \text{ in } A \quad \text{iff} \quad x \xrightarrow{s|t} y \text{ in } \partial A$$



Dual action

Automata A and ∂A $\rightsquigarrow$ two actions $S \curvearrowright X^*$ and $S^* \curvearrowleft X$.

Basic question: What is the relation between these two actions?
What is the relation between G_A and $G_{\partial A}$?

Easy: G_A is finite iff $G_{\partial A}$ is finite

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Connection with Schreier graphs:

The states of $A^{(n)} = A \circ \ldots \circ A$ form a sphere in G_A.

The $(\partial A)^{(n)}$ is the **Schreier graph** $\Gamma_n = \Gamma(G_A, X^n)$.

Bireversible automata

There are two standard operations with automata:

taking inverse: if $s \xrightarrow{x|y} t$ in A, then $s^{-1} \xrightarrow{y|x} t^{-1}$ in $\imath(A)$

taking dual: if $s \xrightarrow{x|y} t$ in A, then $x \xrightarrow{s|t} y$ in $\partial(A)$

A is invertible iff $\imath(A)$ is well-defined
$\partial(A)$ is invertible iff $\imath\partial(A)$ is well-defined

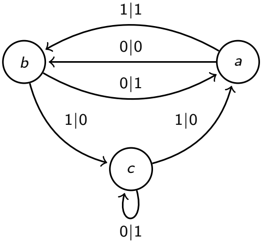
2000 Macedonska, Nekrashevych, Sushchansky: An automaton A is called **bireversible** if all eight automata

$$A,\ \partial(A),\ \imath(A),\ \imath\partial(A),\ \partial\imath(A),\ \partial\imath\partial(A),\ \imath\partial\imath(A),\ \imath\partial\imath\partial(A) = \partial\imath\partial\imath(A)$$

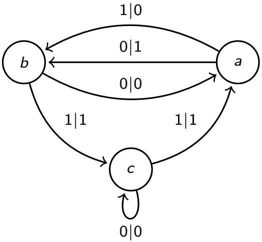
are well-defined (deterministic and complete).

Aleshin and Bellaterra automata

Aleshin and Bellaterra automata are the only bireversile automata with 3 states over 2 letters generating infinite automaton groups.



Aleshin automaton



Bellaterra automaton

Vorobets[2]:

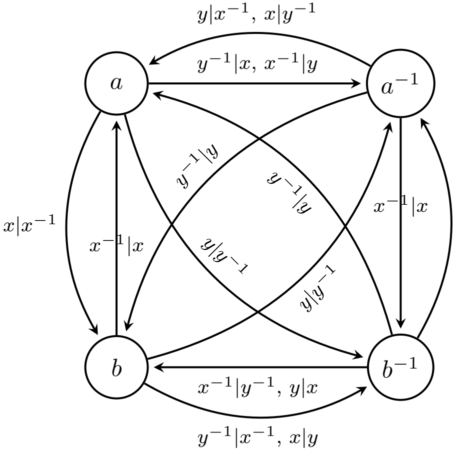
$$G_A = F_3 \text{ free group}$$

$$G_{\partial A} = (\mathbb{Z}_2 \times \mathbb{Z}_2) *_{\mathbb{Z}_2} S_3$$

Muntyan,Nekrashevych,Savchuk:

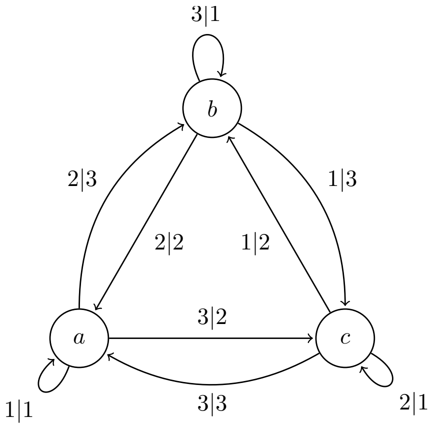
$$G_B = C_2 * C_2 * C_2$$

One more example



Bondarenko-Kivva: $A \cong \partial(A) \cong \imath(A)$ and $G_A = F_2$.

Infinitely presented example



Bondarenko-D'Angeli-Rodaro: $A \cong \partial A \cong \imath(A)$ and $G_A = \mathbb{Z}_3 \wr \mathbb{Z}$

Algebraic object behind duality

We can associate a f.p. group with an automaton $A = (S, X, \tau, \pi)$:

$$\Gamma_A = \langle S, X \mid sx = yt \text{ for each arrow } s \xrightarrow{x|y} t \text{ in } A \rangle.$$

$$\text{certain relations in } \Gamma_A \;\leftrightarrow\; \text{transitions in } A$$

$$s_1x_1x_2x_3 = y_1y_2y_3s_4 \quad \leftrightarrow \quad s_1 \xrightarrow{x_1|y_1} s_2 \xrightarrow{x_2|y_2} s_3 \xrightarrow{x_3|y_3} s_4$$

A is well-defined: $SX \subset XS$ in Γ_A.

$\imath(A)$ is well-defined: $S^{\pm 1}X \subset XS^{\pm 1}$ in Γ_A.

Bireversibility: $S^{\pm 1}X^{\pm 1} = X^{\pm 1}S^{\pm 1}$ in Γ_A. Therefore, we have **group factorization**

$$\Gamma_A = \langle S \rangle \cdot \langle X \rangle$$

Connection with square complexes

2005 Glasner-Mozes: automata vs square complexes

$$\Delta_{\mathrm{A}} = \left\{ \begin{tikzcd} \bullet \arrow[r, "x"] & \bullet \\ \bullet \arrow[u, "s"] \arrow[r, "y"'] & \bullet \arrow[u, "t"'] \end{tikzcd} \quad \text{for each arrow } s \xrightarrow{x|y} t \text{ in A} \right\}.$$

The complex Δ_{A} is a directed VH square complex with one vertex.

$$\pi_1(\Delta_{\mathrm{A}}) = \langle S, X \mid sx = yt \;\; \text{for each arrow } s \xrightarrow{x|y} t \text{ in A} \rangle.$$

Characterizations of bireversibility

Proposition (Follows from Wise, Burger-Mozes, Glasner-Mozes)

Let A *be a finite automaton. The following statements are equivalent:*

1. A *is bireversible;*
2. 2*-cells of* Δ_A *form a* 4*-way deterministic tileset;*
3. *the link of a unique vertex of* Δ_A *is a complete bipartite graph;*
4. Δ_A *is non-positively curved;*
5. *the universal cover of* Δ_A *is a direct product of two trees.*

Properties of $\pi_1(\Delta_{\mathrm{A}})$

Bireversible property: $S^{\pm 1}X^{\pm 1} = X^{\pm 1}S^{\pm 1}$ in $\pi_1(\Delta_{\mathrm{A}})$.

Proposition (Follows from Bridson-Wise)

Let A *be a bireversible automaton. Then:*

1. $\pi_1(\Delta_{\mathrm{A}})$ *is a torsion-free CAT(0) group.*
2. *The subgroups* $\langle S \rangle$ *and* $\langle X \rangle$ *of* $\pi_1(\Delta_{\mathrm{A}})$ *are free groups.*
3. $\pi_1(\Delta_{\mathrm{A}})$ *has exact factorization by* $\langle S \rangle$ *and* $\langle X \rangle$*.*

Every element $\gamma \in \pi_1(\Delta_A)$ can be uniquely written in the **normal forms** $\gamma = gv$ and $\gamma = uh$ for $g, h \in \langle S \rangle$ and $v, u \in \langle X \rangle$. The correspondence between these two normal forms is given by the automaton A^*:

$$gv = uh \text{ in } \pi_1(\Delta_A) \quad \text{if and only if} \quad g \xrightarrow{v|u} h \text{ in } A^*.$$

Automaton groups of bireversible automata

Proposition (Bondarenko-Kivva)

Let A be a bireversible automaton. Then

$$G_A \cong F_S/K \quad \text{and} \quad G_{\partial A} \cong F_X/K_\partial,$$

where K and K_∂ are the maximal normal subgroups of $\pi_1(\Delta_A)$ that are contained in $F_S = \langle S \rangle$ and $F_X = \langle X \rangle$ respectively.

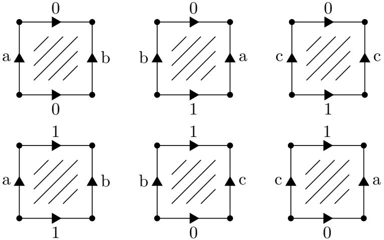
$G_A, G_{\partial A}$ are finite iff $\pi_1(\Delta_A)$ is virtually $F_n \times F_m$ (reducible lattice in the product of two trees)

$\pi_1(\Delta_A)$ is just-infinite $\Rightarrow$ G_A and $G_{\partial A}$ are free groups

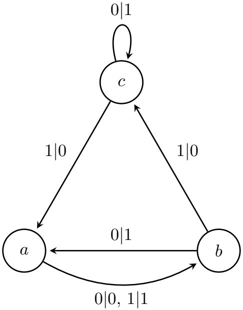
Question: What is the relation between G_A and $G_{\partial A}$?

The transition group: $T_A = \pi_1(\Delta_A)/K \times K_\partial = G_A \cdot G_{\partial A}$.

Aleshin automaton and Wise's example



Wise's complex



Aleshin automaton

2017 Bondarenko-Kivva: $\pi_1(\Delta_A)$ is non-residually finite

Affine action on the ring of formal power series

Let F be a finite field, $F(t)$ rational function field, $F[[t]]$ formal power series ring.

The polynomial ring $F[t]$ has a natural structure of a regular rooted tree with boundary $F[[t]]$, given by the sequence of ideals

$$(1) \supset (t) \supset (t^2) \supset (t^3) \supset \dots$$

The affine group

$$\mathrm{Aff}(F[[t]]) = \{\pi_{a,b} : x \mapsto ax + b,\ a \in F[[t]]^*, b \in F[[t]]\}$$

acts on this tree by automorphisms (self-similar group).

Then: $\mathrm{Aff}(F[[t]]) \cap \mathrm{FAut(F[t])} = \mathrm{Aff}(F(t) \cap F[[t]])$

Example: the lamplighter group

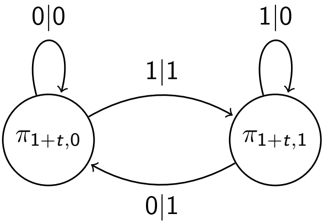
Let $f(t) = 1 + t \in \mathbb{F}_2[t]$. Then

$$\begin{aligned}
\pi_{1+t,0}(0 + a_1 t + \dots) &= 0 + t[(1+t)(a_1 + a_2 t + \dots)] \\
\pi_{1+t,0}(1 + a_1 t + \dots) &= 1 + t[(1+t)(a_1 + a_2 t + \dots) + 1] \\
\pi_{1+t,1}(0 + a_1 t + \dots) &= 1 + t[(1+t)(a_1 + a_2 t + \dots)] \\
\pi_{1+t,1}(1 + a_1 t + \dots) &= 0 + t[(1+t)(a_1 + a_2 t + \dots) + 1]
\end{aligned}$$

$$\begin{aligned}
G_{A_f} &= \langle \pi_{1+t,0}, \pi_{1+t,1} \rangle \\
&= \mathbb{F}_2[(1+t)^{\pm 1}] \rtimes \langle 1 + t \rangle \\
&= \mathbb{Z}_2 \wr \mathbb{Z},
\end{aligned}$$

the lamplighter group

A_f is **not bireversible**



History

2005 Silva-Steinberg: consider $H[[t]]$ for a finite abelian H, $f(t) = 1 - t \in H[[t]]$, $G_{A_f} = H \wr \mathbb{Z}$

2006 Bartholdi-Sunik:

- polynomials over $\mathbb{Z}/n\mathbb{Z}$, $G_A = (\mathbb{Z}/n\mathbb{Z})^d \wr \mathbb{Z}$
- the ring $\mathbb{Z}_n$ of n-adic integer, multiplication by m, G_A is the Baumslag-Solitar group $B(1, m)$

2006 Bartholdi-Neuhauser-Woess: consider a finite commutative ring R, $l_i \in R$ satisfying certain conditions, an automaton group

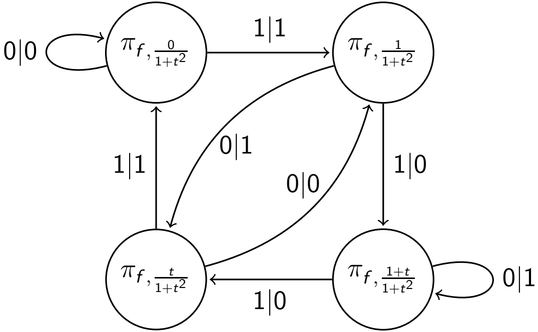
$$\Gamma_d(R) = R[(t + l_1)^{-1}, \dots, (t + l_{d-1})^{-1}, t] \rtimes \langle t + l_1, \dots, t + l_{d-1} \rangle$$

2017 Skipper-Witzel-Zaremsky: for every n a simple group of type F_{n-1} but not of type F_n

Example with a rational function

Let $f(t) = \frac{1+t+t^2}{1+t^2} \in \mathbb{F}_2(t)$. Then A_f is bireversible and

$G_{A_f} = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ and $G_{\partial A_f} = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$.



The automaton A_f

Let $f(t) = \frac{p(t)}{q(t)} \in F(t)$ be non-constant. We assume

$$\begin{array}{lclcl} (q(t),t)=1 & \Rightarrow & f \in F[[t]] & \text{and} & \pi_{f,0} \curvearrowright F[[t]] \\ (p(t),t)=1 & \Rightarrow & f^{-1} \in F[[t]] & \text{and} & \pi_{f,0} \text{ is invertible} \end{array}$$

Let A_f be the automaton for $\pi_{f,0}$. The states of A_f are $\pi_{f,b}$ for $b \in B$, where B is a finite additive subgroup of $F(t)$.

Proposition

1) A_f is bireversible iff $\deg p(t) = \deg q(t)$

2) $G_{A_f} = B[f, f^{-1}] \rtimes \langle f \rangle \cong (\mathbb{Z}/p\mathbb{Z})^m \wr \mathbb{Z}$ *for some* $m \geqslant 1$

Question: What is the dual action? $\partial A_f \stackrel{?}{=} A_{g?}$

Extending the construction

Fix $g(t) \in F(t)$. The field extension $F(t) \supset F(g(t))$ is finite of degree $d = \deg g(t)$, here $1, t, \ldots, t^{d-1}$ is a basis. Then every $f(t) \in F(t)$ can be written as

$$f(t) = r_0(g(t)) + t \cdot r_1(g(t)) + \ldots + t^{d-1} r_{d-1}(g(t)) =$$
$$= \sum_{n \geqslant -k} a_n(t) g^n(t), \text{ where } a_n(t) \in F_{<d}[t].$$

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$R_g = \{f(t) \in F(t) : f(t) = \sum_{n \geqslant 0} a_n(t) g^n(t)\}$ is a subring of $F(t)$.

We can consider **the action of $\mathrm{Aff}(R_g)$ on the tree T_g** given by the ideals $(g^n(t)), n \geqslant 0$.

Automata $A_{(f,g)}$

Let $f(t) = \frac{p_1(t)}{q_1(t)}, g(t) = \frac{p_2(t)}{q_2(t)} \in F(t)$ be such that $f(t) \in R_g$. We can consider the action of $f(t)$ on the tree T_g by multiplication. Let $A_{(f,g)}$ be the corresponding automaton.

If $g(t) = t$ then $A_{(f,g)} = A_f$ is just standard automaton for $f(t)$.

Theorem

1) $A_{(f,g)}$ is bireversible if and only if p_1, p_2, q_1, q_2 are pairwise coprime and either $\deg p_1(t) = \deg q_1(t)$ or $\deg p_2(t) = \deg q_2(t)$.

2) $G_{A_{(f,g)}}$ is a lamplighter group.

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Inverse & Dual: $\imath(A_{(f,g)}) = A_{(f^{-1},g)}$ and $\partial(A_{(f,g)}) = A_{(g^{-1},f^{-1})}$

Eight automata correspond to the pairs $(f^{\pm 1}, g^{\pm 1})$ and $(g^{\pm 1}, f^{\pm 1})$.

Transition group of $A_{(f,g)}$

Let $f(t), g(t) \in F(t)$ and consider two subsets of $\mathrm{Aff}(F(t))$:

$$X_f = \{\pi_{f,a}, a \in S_f\} \quad \text{and} \quad X_g = \{\pi_{g,b}, b \in S_g\},$$

where $S_{\frac{p(t)}{q(t)}} = \left\{ a(t)/q(t) : a(t) \in F[t], \deg a(t) < \deg \frac{p(t)}{q(t)} \right\}.$

The automaton $A_{(f,g)}$ has the states X_f over the alphabet X_g,

$$\pi_{f,a} \xrightarrow{\pi_{g,b}|\pi_{g,b'}} \pi_{f,a'} \quad \text{iff} \quad \pi_{f,a} \circ \pi_{g,b} = \pi_{g,b'} \circ \pi_{f,a'}.$$

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Then

$$\begin{aligned} G_A &= \langle X_f \rangle = S_f[f, f^{-1}] \rtimes \langle f \rangle \\ G_{\partial A} &= \langle X_g \rangle = S_g[g, g^{-1}] \rtimes \langle g \rangle \\ T(A_{(f,g)}) &= \langle X_f, X_g \rangle = \langle X_f \rangle \langle X_g \rangle = G_A \cdot G_{\partial A} \\ &= \left(S_f[f, f^{-1}] + S_g[g, g^{-1}]\right) \rtimes \langle f, g \rangle \end{aligned}$$

The rings of S-integers

Let S be a finite set of places of $F(t)$ and $\mathcal{O}_S$ the ring of S-integers of $F(t)$. We can find $g(t) \in F(t)$ such that

$$\mathcal{O}_S \cap S_g[g, g^{-1}] = \{0\} \quad \Rightarrow \quad \mathrm{Aff}(\mathcal{O}_S) \cap \mathrm{Aff}(S_g[g, g^{-1}]) = \{id\}$$

Theorem

The group $\mathit{Aff}(\mathcal{O}_S)$ can be realized by a bireversible automaton

Corollary: Bireversible automata A such that G_A is finitely presented, while $G_{\partial A}$ is not.

Corollaries

Corollary: A basic construction of

- a family of bireversible automata
- irreducible lattices in the product of two trees
- non-residually finite CAT(0) groups
- directed VH square complexes

Question: What is the nature of $\pi_1(\Delta_{A_{(f,g)}})$?

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- a family of bireversible automata
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Question: What is the nature of $\pi_1(\Delta_{A_{(f,g)}})$?

Conjecture: Groups generated by bireversible automata are linear

Thank You for attention!