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Inverse and Partial Automaton Semigroups

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Why study inverse automaton semigroups?

Conjecture (Steinberg 2015)

There is an automaton group with PSPACE-complete word problem.

- Straight-forward guess and check algorithm:
  (uniform) word problem for automaton (semi)groups is in PSPACE.

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$\rightsquigarrow$ inverse automaton semigroups are useful to obtain results for decision problem

Why study partial automata?

- Groups arise from bijective maps
  (Cayley Theorem)
- Inverse semigroups arise naturally from partial one-to-one maps
  (Preston-Vagner Theorem)
- Automaton Structures:
  - complete, invertible automata generate groups
  - partial, invertible automata generate inverse semigroups

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$\rightsquigarrow$ partial, invertible automata are a natural way to present inverse automaton semigroups

Why restrict automaton semigroups to complete automata?

Why study partial automata? (2)

Reminder: Wang tile $c_w \overset{c_n}{\underset{c_s}{\square}} c_e$, tileset finite set of Wang tiles

Theorem (Lukkarila 2009)

For every Turing machine M, one can compute a 4-way-deterministic Wang tileset T s.t.

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Input: *a partial, invertible, bi-reversible automaton*

Question: *does it generate a finite semigroup?*

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Warning: does not show undecidability of finiteness problem for inverse automaton semigroups!

Example: B_2

Brandt semigroup $B_2 = \langle p, q \mid pqp = p, qpq = q, p^3 = p^2 = q^2 = q^3 \rangle = \{p, q, pq, qp, 0\}$

$= \text{Synt}\left((pq)^+\right) \simeq \text{Synt}\left((ab)^+\right)$

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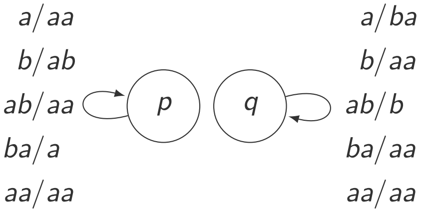
Complete automaton



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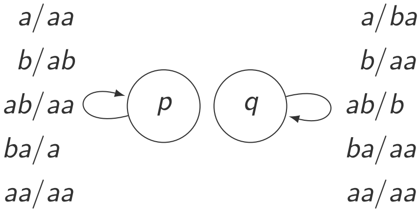

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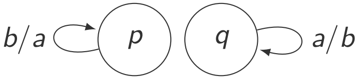

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$qpq = q : a^n \mapsto b^n$

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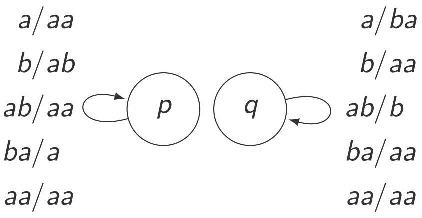

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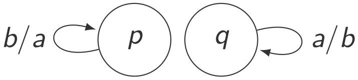

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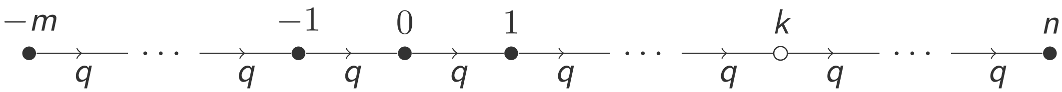
$\rightsquigarrow$ more concise presentation!

Example: Free monogenic inverse monoid

- Every element of the free monogenic inverse monoid is characterized



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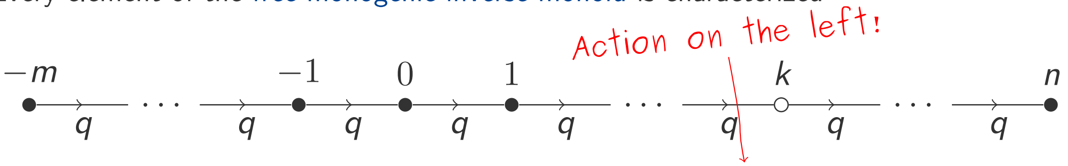

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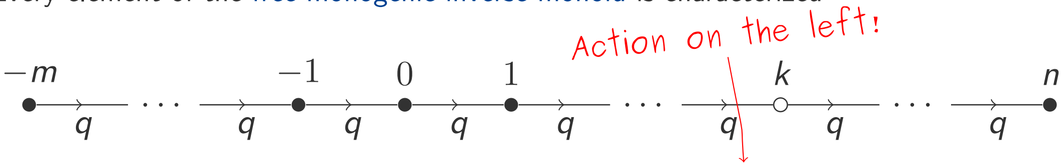
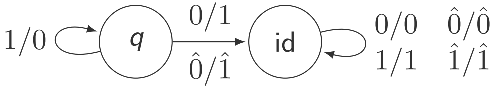


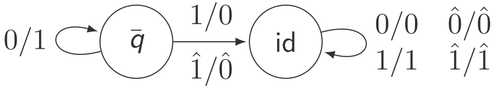
by m, n and k with $-m \leq k \leq n$ and can be written as $\bar{q}^{n-k} q^{m+n} \bar{q}^{m}$

- Claim: The following modification of the adding machine generates the free monogenic inverse monoid.

Automaton



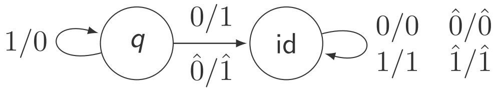
Inverse



(based on a similar construction by Olijnyk, Sushchansky and Slupik)

Proof

Automaton



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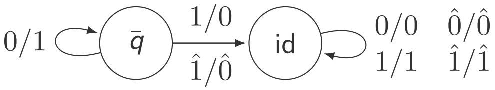

Goal: $\boldsymbol{q} = \bar{q}^{n-k} q^{m+n} \bar{q}^{m} \neq \bar{q}^{n'-k'} q^{m'+n'} \bar{q}^{m'} = \boldsymbol{q'}$ if $m \neq m'$, $n \neq n'$ or $k \neq k'$

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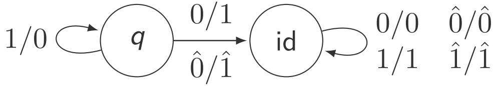

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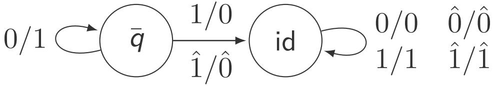

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$$\boldsymbol{q} \circ \mathrm{bin}(0) = \mathrm{bin}(-m + (m+n) - (n-k)) = \mathrm{bin}(k) \neq \mathrm{bin}(k') = \boldsymbol{q'} \circ \mathrm{bin}(0)$$

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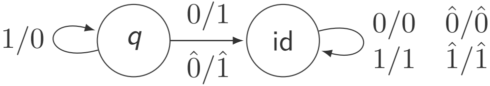

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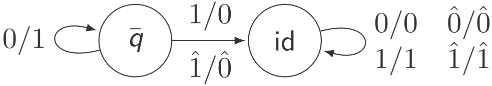

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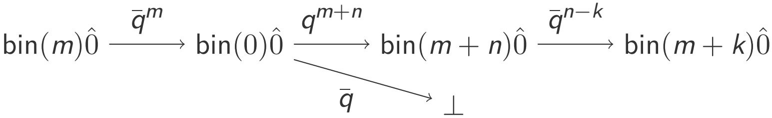

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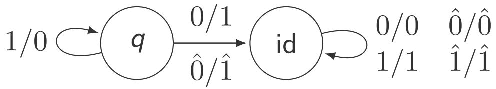

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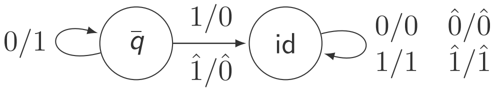

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3. $k = k'$, $m = m'$, $n < n'$:

$$\mathrm{bin}(-1-n)\hat{1} \xrightarrow{\bar{q}^{m}} \mathrm{bin}(-1-n-m)\hat{1} \xrightarrow{q^{m+n}} \mathrm{bin}(-1)\hat{1} \xrightarrow{\bar{q}^{n-k}} \mathrm{bin}(-1-n+k)\hat{1}$$

$$\mathrm{bin}(-1)\hat{1} \xrightarrow{q} \bot$$

Questions

1. Does the class of partial automaton semigroups coincide with the class of complete automaton semigroups? Of course: Every complete automaton is in particular a partial one.

2. Is every inverse automaton semigroup generated by an invertible (partial) automaton?

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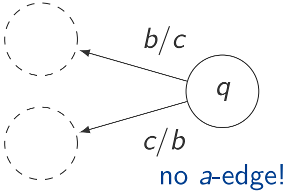
1. Does the class of partial automaton semigroups coincide with the class of complete automaton semigroups? Of course: Every complete automaton is in particular a partial one.
   $\rightsquigarrow$ We don't know! But...
2. Is every inverse automaton semigroup generated by an invertible (partial) automaton?
   $\rightsquigarrow$ Yes! (D'Angeli, Rodaro, W. arXiv 2018)

Adjoining a zero

Reminder: zero of a semigroup $\forall s : 0s = s0 = 0$

- take partial automaton with alphabet $\Sigma = \{a, b, c, \dots\}$ and states $Q = \{p, \dots\}$
- add new state 0 and new letter $\perp$
- make automaton complete:



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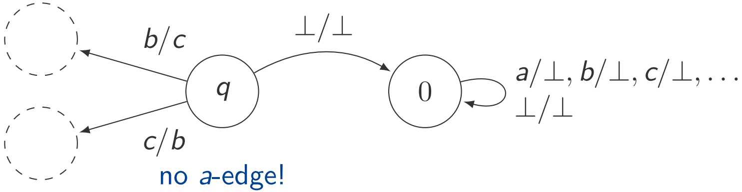

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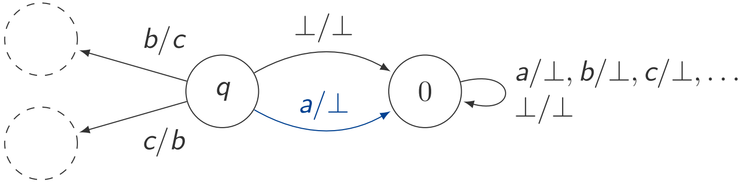

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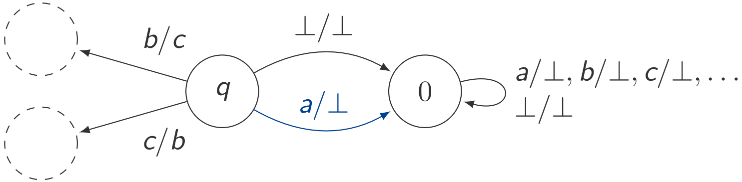

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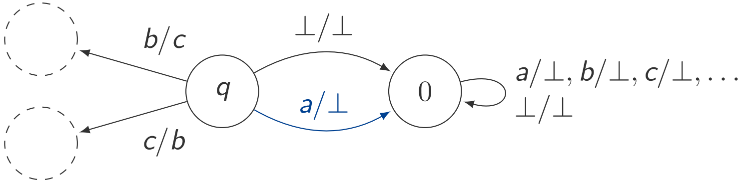

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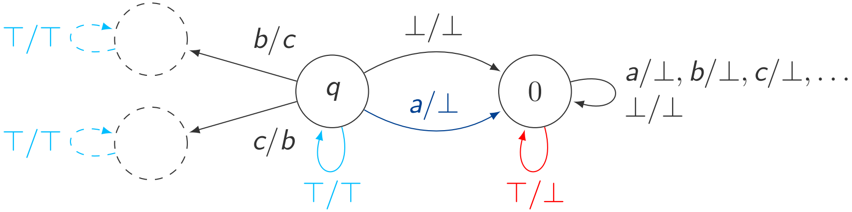

- Does this adjoin a new zero? $S \rightsquigarrow S^0$? $\rightsquigarrow$ In general: No
- add another new letter $\top$

Using a zero

Construction yields:

Proposition (D'Angeli, Rodaro, W. 2017/arXiv 2018)

S partial automaton semigroup $\Longrightarrow$ S^0 complete automaton semigroup

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Theorem (D'Angeli, Rodaro, W. arXiv 2018)

S: partial automaton semigroup
Then: S contains zero $\implies$ S complete automaton semigroup

Proof construction: Assume zero is a state, add (single) new letter and make automaton complete (in the same way as our first attempt above)

No zero

What about automaton semigroups without a zero?

No zero

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Open Problem (Cain 2009)

S^0 complete automaton semigroup $\stackrel{?}{\Longrightarrow}$ S complete automaton semigroup

No zero

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Open Problem (Cain 2009)

S^0 complete automaton semigroup $\overset{?}{\Longrightarrow}$ S complete automaton semigroup

If positive answer: S partial automaton semigroup $\Longrightarrow$ S^0 complete automaton semigroup

$\Longrightarrow$ S complete automaton semigroup

"positive answer" $\Longrightarrow$ "classes coincide"

“positive answer” $\Longleftarrow$ “classes coincide”

What about the other direction?

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What about the other direction?

Theorem (D’Angeli, Rodaro, W. arXiv 2018)

S^0 *partial automaton semigroup* $\Longrightarrow$ S *partial automaton semigroup*

Thus: If classes coincide, problem has positive answer.

Corollary

“positive answer” $\Longleftrightarrow$ *“classes coincide”*

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S^0 partial automaton semigroup $\Longrightarrow$ S partial automaton semigroup

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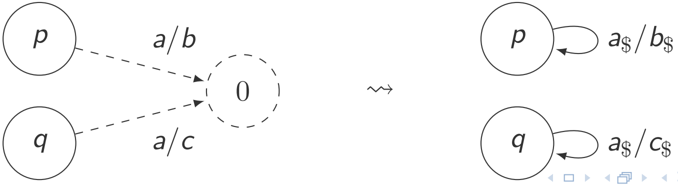
"positive answer" $\Longleftrightarrow$ "classes coincide"

Proof construction (for theorem): Remove zero state(s) with all ingoing transitions

Problem:



Inverse Semigroups and Invertible Automata

Proposition (Cain 2009)

A group is an automaton group if and only if it is an automaton semigroup.

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Every inverse automaton semigroup is generated by a partial, invertible automaton.

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Proof: similar to Cain's proof, uses variation of Preston-Vagner Theorem

Theorem

S: inverse semigroup of partial maps $X \to_p X$

Then: all $\varphi_s : \bar{s}(X) \to s(X)$ *are one-to-one and* $S \to I_X, s \mapsto \varphi_s$ *is injective.*

$$\bar{s}(x) \mapsto s\bar{s}(x)$$

Open Problems

- Do the classes of partial automaton semigroups and of complete automaton semigroups coincide?

  Negative answer is difficult to show: one needs to prove that a partial automaton semigroup is **not** a complete one.
- What is the complexity of the word problem for automaton groups (uniform/non-uniform)? (Steinberg Conjecture)

  known for inverse automaton semigroups
- Is the finiteness problem for inverse automaton semigroups (and groups) decidable?

  strengthened version for invertible automata known

Thank you!