## Schreier graphs of self-similar groups and subshifts of low complexity

Tatiana Nagnibeda<br>University of Geneva<br>joint works with R. Grigorhchuk, D. Lenz, A. Perez, D. Sell

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Spectral graph theory wants to understand how the spectra of various operators defined on (functions on) the graph are related to the geometry of the graph.

There are many possible variations on the theme of the famous question:

Can one hear the shape of a drum? (M. Kac)
A graph 「 $\rightarrow$ the adjacency matrix, the discrete laplacian OR the Markov operator $M=$ transition matrix of the simple random walk on the graph.

An important class of examples: Cayley graphs of (infinite) finitely generated groups. $(G, S) \rightarrow \operatorname{Cay}(G, S)$. (we assume $S=S^{-1}$ )
The Markov operator on the Cayley graph can then be understood as

$$
M\left(=M_{S}\right)=\frac{1}{|S|} \sum_{s \in S} s: I^{2}(G) \curvearrowleft
$$

We can also think of $M$ as an element of the group algebra and consider its images in different representations. The one above (the most classical one) is the left regular representation $\pi: G \rightarrow I^{2}(G)$.

Recently much attention focused on quasi-resular representations of type $\pi_{H}: G \rightarrow I^{2}(G / H)$ where $H<G$ : the operator $\pi_{H}(M)$ is then the Markov operator of the simple random walk on the graph $\operatorname{Sch}(G, H, S)$;
or Koopman representations of type $\pi_{X}: G \rightarrow L^{2}(X, m)$ if we have a measure preserving action of $G$ on a probability space $(X, m)$.

Given a finitely generated group $G$ and/or a group action $G \curvearrowright T$ we are intersted in the spectra of Markov operators on the corresponding Cayley and Schreier graphs.

$$
\begin{gathered}
\operatorname{Mf}(g)=\frac{1}{|S|} \sum_{s \in S} f(g s), \text { for } f \in I^{2}(\operatorname{Vert}(\Gamma)), g \in \operatorname{Vert}(\Gamma) \\
\operatorname{spec}(M) \subseteq[-1,1]
\end{gathered}
$$

Q.: Can one hear the shape of a group? (Valette, Fujiwara)

No: there are non-isomorphic groups with isomorphic Cayley graphs.

Less obvious example: Spectrum of $\mathbb{Z}^{d}$ with standard generators is $[-1,1]$ for each $d$. Moreover, this is the case of any bipartite Cayey graph of a torsion free amenable group.

- The spectrum is symmetric iff $G$ is bipartitie.
- Kesten: $G$ is amenable if and only if $1 \in \operatorname{spec}(M(G, S))$ for every finite symmetric generating set $S \subset G$.
- The absence of non-trivial idempotents in $C_{r}^{*}(G)$ of a torsion free group $G$ (Kadison-Kaplansky Conjecture, true for amenable groups) implies that the spectrum is an interval.

More examples? Non-amenable examples? Torsion examples?

## Q.: What shape can the spectrum take? What is the spectral measure type?

$M$ being a self-adjoint bounded operator on a Hilbert space, its spectrum is a compact subset of $\mathcal{R}$. It gives rise to a projection-valued "spectral" measure on the spectrum which has three components: pure-point, continuous w.r.to the Lebsgue measure and continous singular w.r.to the Lebesgue measure.

There are examples of regular graphs (non Cayley) with a.c. spectral measure on a union of (countably many) intervals with gaps (Aizenmann-Schenker); with p.p. spectrum on a Cantor set of Lebesgue measure 0 (Bartholdi-Grigorchuk), with a non-zero singular continuous component (Simon)...

But maybe more rigidity for Cayley graphs can be expected.

For Cayley graphs, D. Cartwright and G. Kuhn showed in 1970's that it is possible to get the spectrum a union of two intervals or the union of an interval and one or two points on free products of a finite number of cyclic groups.

## Theorem

(Grigorchuk - Dudko, Grigorchuk - N. - Perez '18) For each $m \geq 2$, there is a continuum of pairwise non quasi-isometric groups such that, for a certain choice of generators,

$$
\operatorname{spec}(M)=\left[-\frac{1}{2^{m-1}}, 0\right] \cup\left[1-\frac{1}{2^{m-1}}, 1\right] .
$$

Hence we obtain here examples of Cayley graphs where the spectrum is a union of two intervals. Moreover these are examples of isopectral non quasi-isometric Cayley graphs.

No examples of Cayley graphs with other shape of spectrum, so far
Q.: How does the spectrum depend on the choice of the generating set $S$ ?

Valette - Béguin - Zuk '97 considered the discrete Heisenberg group of $3 \times 3$ matrices, which has two well-known presentations:
$H_{3}=<x, y:[x,[x, y]]=[y,[y, x /]=1>=$
$\langle x, y, z: z=[x, y],[x, z]:[y, z]=1>$.
The spectrum of the first presentation is $[-1,1]$, while they showed that the spectrum of the second presentation is the interval $[(-1-\sqrt{2}) / 3,1]$.

Can the shape of the spectrum change with the change of the generating set? Can the spectral measure type change?

Lamplighter groups with specific generating sets (so-called Diestel-Leader graphs): the spectrum is the interval $[-1,1]$; the spectral measure is pure point (Grigorchuk-Zuk, Lehner-Woess).

Standard generators?? no eigenvalue in the spectrum (Elek).


Isotropic versus Anisotropic random walks
One can naturally consider a random walk on the graph 「 defined by an arbitrary symmetric probability measure on the set of generators $S$. This corresponds to making a step along an edge labelled $s$ with probability $p(s)$, with the conidtion $\sum_{s \in S} p(s)=1$.
Q.: How does the spectrum depend on the choice of the weights $\{p(s)\}_{s \in S}$ on the generators?

The groups. (A variation of a construction of Bartholdi and Sunic, '00)

Let $d \geq 2$ be an integer, and let $T_{d}$ be the $d$-regular infinite rooted tree.

If $X=\{0,1, \ldots, d-1\}$, then $T_{d}$ can be identified with $X^{*}$.
$G \leq \operatorname{Aut}\left(T_{d}\right)$, transitive on each level of the tree. By continuity, the action naturally extends to an action of $G$ on $\partial T_{d}$ by homeomorphisms. The boundary $\partial T_{d}$ can be identified with $X^{\mathbb{N}}$.


Let $m \geq 1$ be an integer.
Consider $A=\mathbb{Z} / d \mathbb{Z}=\langle a\rangle$ and $B=(\mathbb{Z} / d \mathbb{Z})^{m}$.
Let $\omega=\omega_{0} \omega_{1} \cdots \in \Omega_{d, m}^{\mathbb{N}}=\operatorname{Epi}(B, A)^{\mathbb{N}}$.


$$
\begin{gathered}
a\left(v_{0} v_{1} \ldots v_{n}\right)=\left(v_{0}+1\right) v_{1} \ldots v_{n} \\
b\left(v_{0} v_{1} \ldots v_{n}\right)= \begin{cases}(d-1)^{r} 0 \omega_{r}(b)\left(v_{r+1}\right) v_{r+2} \ldots v_{n} & \text { if } v_{0} v_{1} \ldots v_{n}=(d-1)^{r} 0 v_{r+1} \ldots v_{n} \\
v_{0} v_{1} \ldots v_{n} & \text { otherwise }\end{cases}
\end{gathered}
$$

We consider the groups $\left\{G_{\omega}=\langle A, B\rangle\right\}_{\omega \in \Omega_{d, m}} \leq \operatorname{Aut}\left(T_{d}\right)$ with the generating set $S=A \cup B \backslash\{1\}$.

1. Ex. Grigorchuk's group
$d=2, m=2$.
$A=\mathbb{Z} / 2 \mathbb{Z}, B=(\mathbb{Z} / 2 \mathbb{Z})^{2}$.
$\Omega_{d, m}=\operatorname{Epi}(B, A) \quad \Omega_{2,2}=\left\{\pi_{b}, \pi_{c}, \pi_{d}\right\}$
$G_{\omega}=\langle a, b, c, d\rangle$ acts on the binary tree, with $\omega=\left(\pi_{d} \pi_{c} \pi_{b}\right)^{\mathbb{N}}$.
2. Ex. The Gupta-Fabrykowski group
$d=3, m=1$.
$A=\mathbb{Z} / 3 \mathbb{Z}, B=\mathbb{Z} / 3 \mathbb{Z}$.
$\Omega_{d, m}=\operatorname{Epi}(B, A) \quad \Omega_{3,1}=\left\{\pi_{1}, \pi_{2}\right\}$
$G_{\omega}=\langle a, b\rangle$ acts on the ternary tree, with $\omega=\pi_{1}^{\mathbb{N}}$.
3. Ex: $d=2, m=1,|\Omega|=1, G_{1^{\mathbb{N}}}=D_{\infty}$.

Schreier graphs of the Grigorchuk's group ( $d=2, m=2$,
$\left.\omega=\left(\pi_{d} \pi_{c} \pi_{b}\right)^{\mathbb{N}}, G=\langle a, b, c, d\rangle\right)$ for the action on the levels of the tree:


Schreier graphs of the Gupta-Fabrykowski group $(d=3, m=1$, $\left.\omega=\pi_{1}^{\mathbb{N}}, G=\left\langle a, a^{2}, b, b^{2}\right\rangle\right)$ for the action on the levels of the tree:

$\Gamma_{2}$


## Scaling limit of finite Schreier graphs $\Gamma_{n}$

The limit space of the Gupta-Fabrykowski group is $J\left(z^{3}\left(-\frac{3}{2}+i \frac{\sqrt{3}}{2}\right)+1\right)$.



Infinite Schreier graphs of Grigorchuk's group $(d=2, m=2$, $\left.\omega=\left(\pi_{0} \pi_{1} \pi_{2}\right)^{\mathbb{N}}, G_{\omega}=\langle a, b, c, d\rangle\right)$ for the action on the boundary of the tree:

$\Gamma_{0^{\mathbb{N}}}$

Infinite Schreier graphs of the Gupta-Fabrykowski group
( $\left.d=3, m=1, \omega=\pi^{\mathbb{N}}, G_{\omega}=\left\langle a, a^{2}, b, b^{2}\right\rangle\right)$ for the action on the boundary of the tree:


## Schreier dynamical system.

To each point $\xi$ of the boundary we associate its orbital Schreier graph $\left(\Gamma_{\xi}, \xi\right)$, viewed as a point in the space $\mathcal{G}_{*, S}$ of rooted, oriented, labeled graphs equipped with local convergence.

$$
\begin{array}{rlll}
F: & \partial T_{d} & \rightarrow \mathcal{G}_{*, S} \\
\xi & \mapsto & \left(\Gamma_{\xi}, \xi\right)
\end{array}
$$

## Proposition

- If $\xi=\xi_{0} \xi_{1} \ldots$, the sequence $\left(\Gamma_{\xi_{0} \ldots \xi_{n}}, \xi_{n}\right)$ converges to $\left(\Gamma_{\xi}, \xi\right)$.
- For all $G_{\omega}$ except $d=2, m=1, F$ is injective.
- For all $G_{\omega}, F$ is continuous except for the orbit $G_{\omega} \cdot(d-1)^{\mathbb{N}}$.

Schreier dynamical system $G \curvearrowright \bar{F}$, a closed subspace of $\mathcal{G}_{*, S}$. The action is p.m.p. w.r.t $F_{*} \nu$ with $\nu$ the uniform measure on $\partial T$.

Remark. Spectrum of $\Gamma_{\xi}$ doesn't depend on $\xi$.

## Ends.

Consider the number of ends of each graph. $\Gamma$ is $k$-ended if, for every $\eta \in \Gamma, \Gamma \backslash\{\eta\}$ has $\leq k$ infinite connected components, and $k$ is minimal.

## Proposition

Any $\Gamma_{\xi}$ has either one or two ends:

- $\Gamma_{\xi}$ is two-ended if $\xi \in X^{*}\{0, d-1\}^{\mathbb{N}} \backslash G_{\omega} \cdot(d-1)^{\mathbb{N}}$.
- $\Gamma_{\xi}$ is one-ended otherwise.

We define $\operatorname{Sch}(G):=\overline{F\left(\partial T_{d}\right)}$.

## Theorem

(N., Perez, '17) $\operatorname{Sch}\left(G_{\omega}\right)$ contains one-ended, two-ended and $d$-ended graphs. If $d=2$, the generic case is two ends, if $d \geq 3$, the generic case is one end.

Spectral measure $\nu$ for Grigorchuk's group $(d=2)$


Spectral measure $\nu$ for the Gupta-Fabrykowski group $(d>2)$


## Recall: Theorem

For $d=2$, any $m \geq 1$, and any $G_{\omega}$ with the generating set $S$ as above,

$$
\operatorname{spec}(G)=\operatorname{spec}(\Gamma)=\left[-\frac{1}{2^{m-1}}, 0\right] \cup\left[1-\frac{1}{2^{m-1}}, 1\right] .
$$

## Theorem: Changing the generating set

(Follows from Grigorchuk, Lenz, N., '15; Grigorchuk, Lenz, N., Sell '18)

For $d=2, m \geq 2$ and any $G_{\omega}$ there exists a (minimal) generating set with the spectrum of any infinite Schreier graph 「 a Cantor set of Lebesgue measure zero, purely singular continuous for almost every $\omega$.

The last Theorem is proved via considering anisotropic random walks on infinite Schreier graphs and realizing the corresponding operators as Schroedinger operators on subshifts of low complexity.
Q.: What is the spectrum of the Cayley graph for this generating set?

## Theorem

(Grigorchuk, Lenz, N., '15) Let $M_{w}$ be a weighted laplacian on the Schreier graph $\Gamma=\Gamma_{\xi}$ of a group $G_{\omega}$ with $d=2$. Then there exists a subshift $\Sigma_{\omega}$ such that $M_{w}$ is unitary equivalent to the Schroedinger operator on the subshift $\left(T, \Sigma_{\omega}\right),\left\{H_{\sigma}: I^{2}(\mathbb{Z}) \curvearrowleft\right\}_{\sigma}$ given by two functions $\alpha, \beta: \Sigma_{\omega} \rightarrow \mathbb{R}$ and, for every $u \in I^{2}(\mathbb{Z})$,

$$
\left(H_{\sigma} u\right)(n)=\alpha\left(T^{n-1} \sigma\right) u(n-1)+\alpha\left(T^{n} \sigma\right) u(n+1)+\beta\left(T^{n} \sigma\right) u(n)
$$

For subshifts of low complexity, the following statement is dubbed "Cantor spectrum of Lebesgue measure 0"-theorem:

The spectrum of such operators is absolutely continuous on an interval or a union of two intervals if $\alpha, \beta$ are periodic and, if not, it is a Cantor set of Lebesgue measure 0 ; the spectral measure is $\omega$ - a.s. singular continuous.

## Remarks

1. Periodic $\alpha, \beta$ correspond to isotropic weights $w$, and aperiodic $\alpha, \beta$ - to anisotropic weights $w$ on the generators $S$, in case of subshifts determined by Schreir graphs.
2. If the spinal group is generated by an automaton, then the subshift is substitutinal.

For the Grigorchuk group, the substitution is

$$
\kappa: a \mapsto a c a ; b \mapsto d ; d \mapsto c ; c \mapsto b
$$

The subshift is defined by the fixed point of the substitution $\eta=\lim _{n} \kappa^{n}(a)$ as the subset of $\{a, b, c, d\}^{\mathcal{Z}}$ : it consists of all two-sided sequences whose set of finite subwords coincides with the set of finite subwords of $\eta$.

## What do we mean by low complexity of a subshift?

Sufficient conditions for "Cantor spectrum of Lebesgue measure 0 theorem":

- linear repetitivity ( $\omega$ - negligeble condition). A subshift $(T, \Sigma)$ is called linearly repetitive (LR), if there exists a constant $C>0$ such that any word $v \in \operatorname{Sub}(\Sigma)$ occurs in any word $w \in \operatorname{Sub}(\Sigma)$ of length at least $C|v|$. (Damanik - Lenz)
- Boshernitsan condition ( $\omega$ a.s. condition). A subshift satisfies the Boshernitsan condition (B) if the same condition is satisfied for all $v$ of length $I_{n}$, for a certain increasing sequence $\left\{I_{n}\right\}$. (Beckus Pogorzelski)
- Simple Toeplitz subshifts and their generalizations (all $\omega$ ). (Grigorchuk - Lenz - N. - Sell '18).

In a work with Perez, we define analogous notions of low complexity (linear repetitivity, Boshernitzan, simple Toeplitz) for (non-linear) Schreier graphs with $d \geq 3$, and show that all Schreier graphs of groups $G_{\omega}$ are of low complexity, as in the case of $d=2$.
Q.: Does it have implications on the spectrum type for (anisotropic) random walk?

Recall that by results of Grigorchuk-N.-Perez, the spectrum of the isotropic random walk on these graphs is a Cantor set of Lebesgue measure zero.

Thank you!

