Spectral properties related to spinal groups

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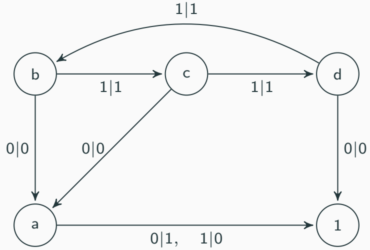
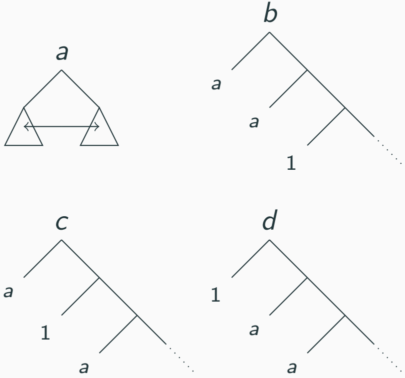
Groups, automata and graphs, TUGraz

Outline

Spinal groups

Spinal groups - Grigorchuk's group





Grigorchuk's group: $G = \langle a, b, c, d \rangle \leq \mathrm{Aut}(X^*), \quad X = \{0, 1\}.$

$$A = \langle a \rangle = \mathbb{Z}/2\mathbb{Z} \qquad B = \langle b, c, d \rangle = (\mathbb{Z}/2\mathbb{Z})^2$$

Spinal groups - Definition

We want to generalize Grigorchuk's group in several ways:

- Action on any regular rooted tree:

$$d \geq 2 \longrightarrow A = \langle a \rangle = \mathbb{Z}/d\mathbb{Z}$$

- More elements in B:

$$m \geq 1 \longrightarrow B = (\mathbb{Z}/d\mathbb{Z})^m$$

Spinal groups - Definition

Let $d \geq 2$ and $X = \{0, 1, \ldots, d-1\}$.

Let $m \geq 1$, $A = \mathbb{Z}/d\mathbb{Z} = \langle a \rangle$ and $B = (\mathbb{Z}/d\mathbb{Z})^m$.

Definition [Bartholdi, Šunić, 2000]

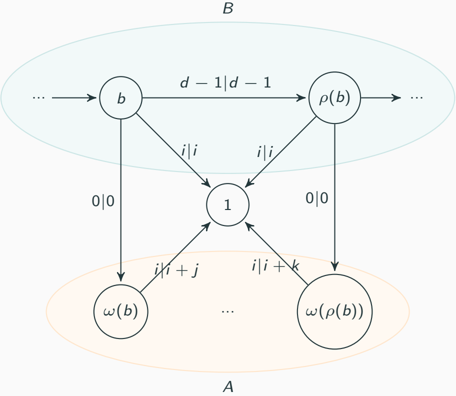
An automaton with states $A \cup B$ and alphabet X defines a **spinal group** if its edges are of these types

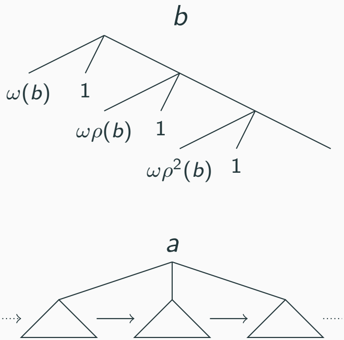
for some epimorphism $\omega : B \to A$ and automorphism $\rho : B \to B$.

$$G = \langle A \cup B \rangle \leq \mathrm{Aut}(X^*)$$

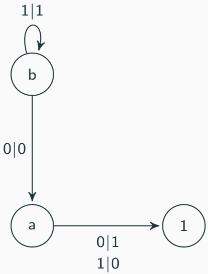
(The definition can be generalized so that, for every d and m, we obtain an uncountable family of groups)

Spinal groups - Examples





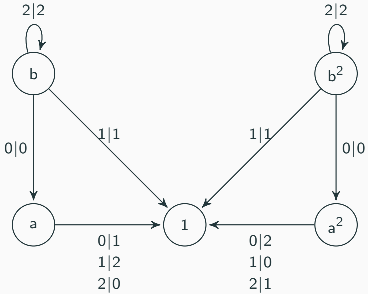
Infinite dihedral



$$D_\infty = \langle a, b \rangle$$

	ω	ρ
$d = 2$		
$m = 1$	$b \mapsto a$	$b \mapsto b$

The Fabrykowski-Gupta group



$$G = \langle a, a^2, b, b^2 \rangle$$

	ω	ρ
$d = 3$	$b \mapsto a$	$b \mapsto b$
$m = 1$	$b^2 \mapsto a^2$	$b^2 \mapsto b^2$

Schreier graphs

Schreier graphs - Definition

Definition

Let G be a group, finitely generated by $S = S^{-1}$, acting on a set Y. We define its **Schreier graph** $\mathrm{Sch}(G, S, Y)$ as the graph given by

- $V = Y$.
- $E = \{(z, sz) \mid z \in Y, s \in S\}$.

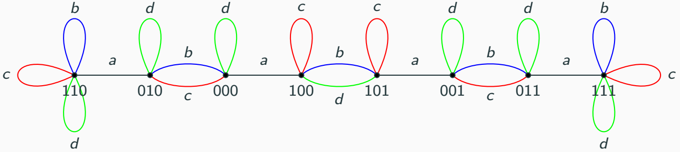
The graph is oriented and edge-labeled by the set S.

For spinal groups, we will always consider $S = (A \cup B) \setminus \{1\}$.

Schreier graphs - Examples

Grigorchuk's group: $G = \langle a, b, c, d \rangle$

$$d = 2 \qquad m = 2 \qquad \begin{array}{c} \omega \\ b \mapsto a \\ c \mapsto a \\ d \mapsto 1 \end{array} \qquad \begin{array}{c} \rho \\ b \mapsto c \\ c \mapsto d \\ d \mapsto b \end{array}$$



$$\Gamma_3 = \mathrm{Sch}(G, S, X^3)$$

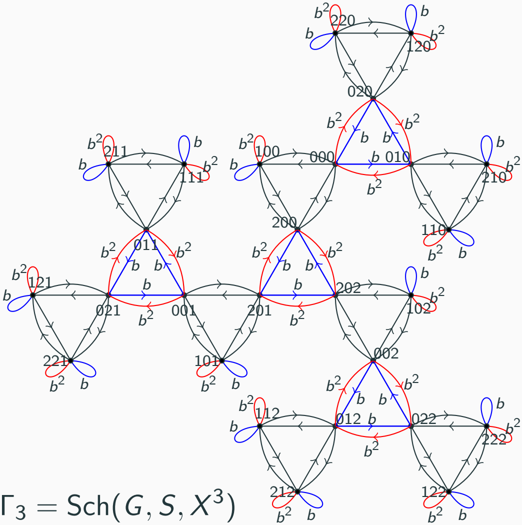
The Fabrykowski-Gupta group: $G = \langle a, a^2, b, b^2 \rangle$

$d = 3$

$m = 1$

ω
$b \mapsto a$
$b^2 \mapsto a^2$

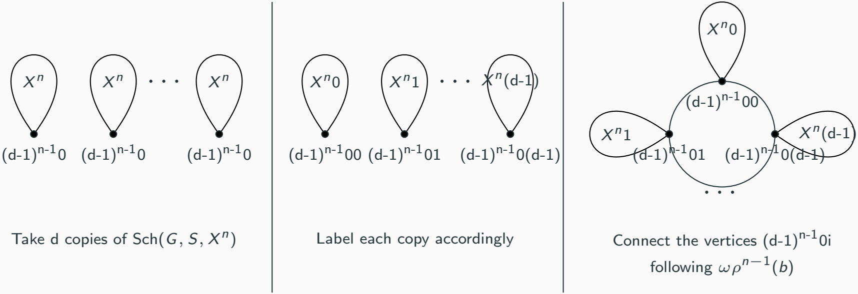
ρ
$b \mapsto b$
$b^2 \mapsto b^2$



$\Gamma_3 = \mathrm{Sch}(G, S, X^3)$

Schreier graphs - Construction

There is a natural recursive way of constructing $\mathrm{Sch}(G, S, X^{n+1})$ from $\mathrm{Sch}(G, S, X^n)$ (similar to Bondarenko's inflation of graphs):



$$\forall v_0 \dots v_{n-1} \in X^n \setminus \{(d-1)^{n-1}0\}, \quad \forall i \in X, \quad \forall s \in S,$$

$$s(v_0 \dots v_{n-1} i) = s(v_0 \dots v_{n-1}) i$$

Schreier graphs - Construction

The action of G can be extended naturally to the boundary $X^{\mathbb{N}}$ of the tree. Orbits are cofinality classes.

For $\xi \in X^{\mathbb{N}}$, the marked graph $(\mathrm{Sch}(G, S, G\xi), \xi)$ is the limit of $(\mathrm{Sch}(G, S, X^n), \xi_0 \dots, \xi_{n-1})$ in the space of rooted graphs.

Definition

A sequence of rooted graphs (Γ_n, v_n) converges to (Γ, v) if

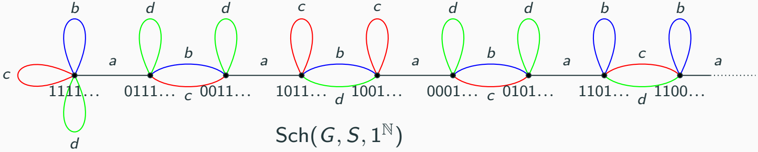
$$\forall r \in \mathbb{N}, \quad \exists N \in \mathbb{N}, \quad \forall n \geq N, \quad B_{v_n}(r) \cong B_v(r).$$

Schreier graphs - More examples

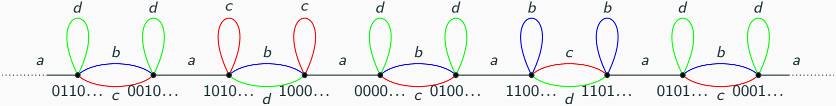
Grigorchuk's group: $G = \langle a, b, c, d \rangle$

$d = 2 \qquad m = 2$

ω	ρ
$b \mapsto a$	$b \mapsto c$
$c \mapsto a$	$c \mapsto d$
$d \mapsto 1$	$d \mapsto b$



$\mathrm{Sch}(G, S, 1^{\mathbb{N}})$



$\mathrm{Sch}(G, S, 0^{\mathbb{N}})$

Schreier graphs - More examples

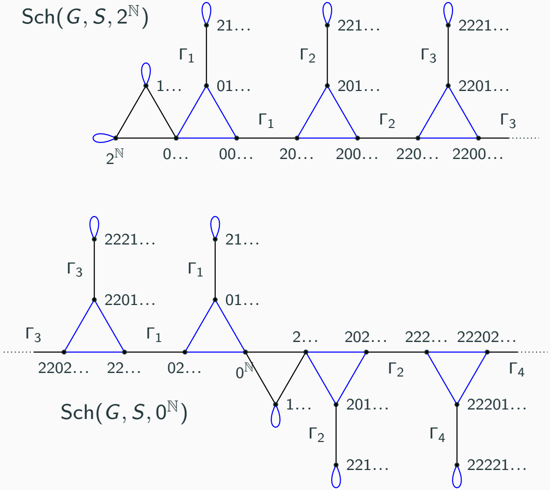
The Fabrykowski-Gupta group: $G = \langle a, a^2, b, b^2 \rangle$

$d = 3$

$m = 1$

ω
$b \mapsto a$
$b^2 \mapsto a^2$

ρ
$b \mapsto b$
$b^2 \mapsto b^2$



Schreier graphs - Space of rooted graphs

Let $\mathcal{G}_{S,*}$ be the space of rooted graphs with edge labels in S. We consider the map

$$\begin{array}{rccc} \mathcal{F}: & X^{\mathbb{N}} & \to & \mathcal{G}_{S,*} \\ & \xi & \mapsto & (\Gamma_\xi, \xi) \end{array}$$

Remarks

- $\mathcal{F}$ is injective.

- $\mathcal{F}$ is continuous everywhere except in the orbit of $(d-1)^{\mathbb{N}}$.

- $\mathcal{F}(X^{\mathbb{N}})$ contains only one and two-ended graphs, but $\overline{\mathcal{F}(X^{\mathbb{N}})}$ contains d-ended graphs as well.

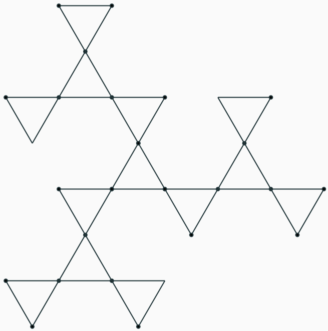
- $\overline{\mathcal{F}(X^{\mathbb{N}})}$ has isolated points iff $d = 2$.

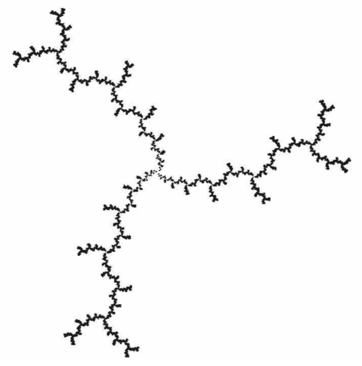
- The growth of Γ_ξ is polynomial of degree $\log_2(d)$ [Bondarenko].

Schreier graphs - Limit spaces

Nekrashevych defined a notion of **limit space** $\mathcal{J}_G$ for a contracting (finite nucleus) automata group G.

We can embed the graphs Γ_n in the plane in a way that they *approximate* $\mathcal{J}_G$:





Spectral properties

Spectral properties - The Markov operator

Definition

Let $\Gamma = (V, E)$ be a k-regular graph.

The Markov operator $M : \ell^2(V) \to \ell^2(V)$ is defined by

$$Mf(v) = \frac{1}{k} \sum_{w \sim v} f(w)$$

In our case:

$$\Gamma_n = \mathrm{Sch}(G, S, X^n) \longrightarrow \begin{array}{c} M_n : \ell^2(X^n) \to \ell^2(X^n) \\ M_n f(w) = \frac{1}{|S|} \sum_{s \in S} f(s^{-1} w) \end{array}$$

$$\Gamma_\xi = \mathrm{Sch}(G, S, G\xi) \longrightarrow \begin{array}{c} M_\xi : \ell^2(G\xi) \to \ell^2(G\xi) \\ M_\xi f(\eta) = \frac{1}{|S|} \sum_{s \in S} f(s^{-1} \eta) \end{array}$$

Spectral properties - Spectrum of M_ξ

We can exploit the self-similar nature of spinal groups in order to compute the spectrum of the Markov operator M on Γ_ξ.

Theorem [Dixmier '77, Proposition 3.4.9]

$$\mathrm{spec}(M_\xi) \subset \overline{\bigcup_{n\geq 0} \mathrm{spec}(M_n)}$$

$$\Gamma_\xi \text{ amenable} \quad \Rightarrow \quad \mathrm{spec}(M_\xi) = \overline{\bigcup_{n\geq 0} \mathrm{spec}(M_n)}$$

Notice: $\mathrm{spec}(M_\xi)$ does not depend on ξ.

Bartholdi and Grigorchuk computed the spectrum for Grigorchuk's group (two intervals), the Fabrykowski-Gupta group (a Cantor set plus a countable set), and other related examples.

Spectral properties - Spectrum of M_ξ

We have

$$M_n = \frac{1}{|S|}(A_n + B_n)$$

with

$$A_n = \begin{bmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix}$$

$$B_n = \begin{bmatrix} d^{m-1}A_{n-1} + d^{m-1} - 1 & & & & \\ & d^m - 1 & & & \\ & & \ddots & & \\ & & & d^m - 1 & \\ & & & & B_{n-1} \end{bmatrix}$$

Spectral properties - Spectrum of M_ξ

We use the Schur complement method

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B)$$

to find a relation between $\mathrm{spec}(M_n)$ and $\mathrm{spec}(M_{n-1})$:

$$z \in \mathrm{spec}(M_n(t)) \Longleftrightarrow z' \in \mathrm{spec}(M_{n-1}(t'))$$

$$z' = \Phi_1(t, z), \quad t' = \Phi_2(t, z)$$

Solving this recurrence allows to find $\mathrm{spec}(M_n)$ explicitly.

Spectral properties - Spectrum of M_ξ

Theorem [Grigorchuk, Nagnibeda, P.]

$$\mathrm{spec}(M_n) = \{1, \lambda_0\} \cup \psi^{-1}\left(\bigcup_{k=0}^{n-2} F^{-k}(0)\right)$$

$$\mathrm{spec}(M_\xi) = \{1, \lambda_0\} \cup \psi^{-1}\left(\overline{\bigcup_{n\geq 0} F^{-n}(0)}\right)$$

where $F(x) = x^2 - d(d-1)$ and ψ is a quadratic map.

If $d = 2$, $\mathrm{spec}(M_\xi) = [-\frac{1}{2^{m-1}}, 0] \cup [1 - \frac{1}{2^{m-1}}, 1]$.

If $d \geq 3$, $\mathrm{spec}(M_\xi)$ is a Cantor set plus a countable set of points.

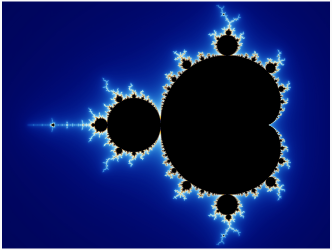
Notice: $\mathrm{spec}(M_\xi)$ depends only on d and m.

Spectral properties - Spectrum of M_ξ

spec(M_ξ) is obtained as the preimage by the quadratic map ψ of the **Julia set** of $F(x) = x^2 - d(d-1)$.

Julia set of $F(x) = x^2 - 2$

Julia set of $F(x) = x^2 - 6$



Mandelbrot set

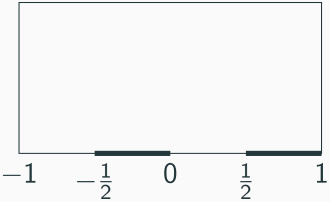
Spectral properties - Spectral measure

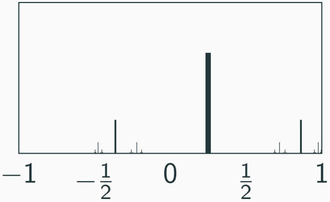
The **empirical spectral measure** ν of $\{\Gamma_n\}_n$ is the weak limit of the counting measures ν_n on Γ_n.

Theorem [Grigorchuk, Nagnibeda, P.]

If $d = 2$, ν is absolutely continuous with respect to the Lebesgue measure.

If $d \geq 3$, ν is concentrated in the set of eigenvalues of M_n.





Spectral properties - Spectrum of M_G

We may also consider the Markov operator on Cay(G, S), the Cayley graph of G:

$$M_G : \ell^2(G) \to \ell^2(G)$$
$$M_G f(g) = \frac{1}{|S|} \sum_{s \in S} f(s^{-1}g)$$

Theorem [Hulanicki]

G amenable $\Rightarrow$ spec$(M_\xi) \subset$ spec(M_G) for every $\xi \in X^{\mathbb{N}}$.

Theorem [Grigorchuk, Dudko; Grigorchuk, Nagnibeda, P.]

If $d = 2$, spec$(M_\xi) =$ spec(M_G) for every $\xi \in X^{\mathbb{N}}$.

Spectral properties - Final remarks

Remark

If $d = 2$, Cayley and Schreier graphs have the same spectrum.

Corollary

There are uncountably many groups whose spectrum is the union of two intervals.

Corollary

There are uncountably many pairwise non quasi-isometric isospectral groups.

Thank you!