STURMIAN SUBSTITUTIONS, CUTTING PATHS AND THEIR PROJECTIONS

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Words

An alphabet $\mathcal{A}$ is a finite set of elements that are called letters. We take $\mathcal{A} = \{0, 1\}$.

A word $u$ is a function $u$ from a finite or infinite block of integers to $\mathcal{A}$. If this block of integers contains negative numbers we call $u$ a central word. If a word $u$ is finite, we denote by $|u|$ the number of letters in $u$, and by $|u|_a$ the number of occurrences of the letter $a$ in $u$.

A word $u$ is called balanced if $|v|_0 - |w|_0 < 2$ for all subwords $v, w$ of equal length. A finite word $u$ is called strongly balanced if $u^2$ is balanced. Here $u^2$ is the concatenation of $u$ with $u$.

A strongly balanced word is called a Christoffel word when it is smaller than each of its shifts in the lexicographic order (the zeros are placed as far to the left as possible).
Central progressions $w_u$

**Definition.** Let $u = u(0)\ldots u(m - 1)$ be a strongly balanced finite word, containing both zeros and ones, with $\gcd(|u|_0, |u|_1) = 1$. The *cutting path* in the $x$-$y$-plane corresponding to $u$ consists of $m + 1$ integer points $p_i$ given by $p_i = (|u(0)\ldots u(i - 1)|_0, |u(0)\ldots u(i - 1)|_1)$ for $i = 0, \ldots, m$, connected by line segments of lengths 1.

Draw the line through the origin and the end point of the path, given by $y = \frac{|u|_1}{|u|_0}x$. We project each integer point $p_i$ on the cutting path parallel to this line onto the $y$-axis. By $P(p_i)$ we denote the second coordinate of the projection of $p_i$. It is clear that $P(p_0) = P(p_m) = 0$. 

**Definition.** We define the function $w_u$ as follows. If $P(p_i) = k/|u|_0$ then $w_u(-k) = i$. We say $w_u$ has the number $i$ at position $-k$. We call $w_u$ the *central progression* corresponding to $u$.

Some properties of a central progression $w_u$ corresponding to $u = u(0) \ldots u(m-1)$.

- Its domain is a block of integers of length $m$ of $\mathbb{Z}$ containing 0.

- Its image is the set $\{0, 1, \ldots m - 1\}$.

- There exists a $c \in \mathbb{Z}$ such that if $k$ is in the domain of $w$, then $w(k) \equiv ck \pmod{m}$.

**Example 1.** Let $u = 01001$, then the central progression $w_u$ is given by $w = 20314$. 
Central words $v_w$

Definition. Let $w$ be a central progression. Then the central word $v_w$ is the word that you get by replacing every number in $w$ that is smaller than its right neighbour by 0, and every number that is larger by 1.

Example 2. If $u = 01001$, then $w_u = 20314$, and $v_w = 10\underline{1}01$, where we underlined the letter at position 0.
If $u = 0110101101101$, then $w_u = 11 \ 3 \ 8 \ 0 \ 5 \ 10 \ 2 \ 7 \ 12 \ 4 \ 9 \ 1 \ 6$, and $v_w = 101\underline{0}01\underline{0}010100$. 
**Sturmian substitutions**

A *substitution* $\sigma$ is an application from the alphabet $\mathcal{A} = \{0, 1\}$ to the set of finite words. It extends to a morphism by concatenation, that is, $\sigma(uv) = \sigma(u)\sigma(v)$.

A *fixed point* of a substitution $\sigma$ is an infinite word $u$ with $\sigma(u) = u$.

If $\sigma$ is a substitution, we call

$$M_\sigma = \begin{pmatrix} |\sigma(0)|_0 & |\sigma(0)|_1 \\ |\sigma(1)|_0 & |\sigma(1)|_1 \end{pmatrix}$$

its *incidence matrix*.

A one-sided infinite word is *Sturmian* if it is balanced and not ultimately periodic.

We call a substitution $\sigma$ over two letters *Sturmian* if $\sigma(u)$ is a Sturmian word for every Sturmian word $u$. 
Let $\sigma$ be a Sturmian substitution that has incidence matrix with determinant 1 and a fixed point starting with 0, let $u_n = \sigma^n(0)$, let $w_n = w_{u_n}$ and let $v_n = v_{w_n}$ for $n > 0$.

Example 3. Let $\sigma$ be the substitution defined by $\sigma(0) = 010$, $\sigma(1) = 01$. Then $\sigma$ is a Sturmian substitution with $M_\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. We have

$u_0 = 0$
$u_1 = 010$
$u_2 = 01001010$
$u_3 = 01001010010010101010$

\ldots
This yields the following table of central progressions $w_n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$w_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2 0 1</td>
</tr>
<tr>
<td>2</td>
<td>7 2 5 0 3 6 1 4</td>
</tr>
</tbody>
</table>
| 3   | 20 7 15 2 10 18 5 13 0 8 16 3 →  
|     | → 11 19 6 14 1 9 17 4 12          |
|     | …….            |

We get the following table of central words.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$v_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1 0 0</td>
</tr>
<tr>
<td>2</td>
<td>1 0 1 0 0 1 0 0 0 1 0 0</td>
</tr>
</tbody>
</table>
| 3   | 1 0 1 0 0 1 0 1 0 0 1 0 0 0 1 0 0 →  
|     | → 0 1 0 1 0 0 1 0 0 1 0 0          |
|     | …….            |

Looking at the table above, we notice that if we define a substitution $\tau$ by $\tau(0) = 100$, $\tau(1) = 10$, then $\tau(v_n) = v_{n+1}$ for each $n \geq 0$. 
**Question.** For which substitutions $\sigma$ does there exist a substitution $\tau$ so that $\tau(v_n) = v_{n+1}$ for each $n \geq 0$, and what can we say about $\tau$?

**Theorem.** Let $\sigma$ be a Sturmian substitution with $\sigma(0)$ and $\sigma(1)$ Christoffel words. Then $\tau$ exists, and it is also a Christoffel substitution.

**Example 4.**

- $\sigma : \begin{cases} 0 \rightarrow 00101 \\ 1 \rightarrow 01 \end{cases}$ gives $\tau : \begin{cases} 0 \rightarrow 01011 \\ 1 \rightarrow 011 \end{cases}$

- $\sigma : \begin{cases} 0 \rightarrow 0010101 \\ 1 \rightarrow 01 \end{cases}$ gives $\tau : \begin{cases} 0 \rightarrow 0110111 \\ 1 \rightarrow 0111 \end{cases}$

- $\sigma : \begin{cases} 0 \rightarrow 0101011 \\ 1 \rightarrow 01011 \end{cases}$ gives $\tau : \begin{cases} 0 \rightarrow 000101 \\ 1 \rightarrow 001 \end{cases}$
From now on, assume $\sigma$ is a Sturmian substitution that has incidence matrix with determinant 1, a fixed point starting with 0 and is NOT a Christoffel substitution. Put $M_\sigma = \begin{pmatrix} a & b \\ c + ag & d + bg \end{pmatrix}$ with $a + b > c + d$.

Denote by $e$ the number of values left of the 0 position in $w_{\sigma(0)} = w_1$, by $f$ the number of values left of the 0 position in $w_{\sigma(1)}$, by $p$ the number of zeros in $v_1$ left of the underlined letter, and set $r = e + b(f - p - eg)$.

**Example 5.** Let $M_\sigma = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ and $\sigma(0) = 01001$, $\sigma(1) = 010010101001$. Then $g = 2$, $w_{\sigma(0)} = 2 ~ 0 ~ 3 ~ 1 ~ 4$, $v_1 = 10101$ and $w_{\sigma(1)} = 9 ~ 2 ~ 7 ~ 0 ~ 5 ~ 10 ~ 3 ~ 8 ~ 1 ~ 6 ~ 11 ~ 4$. Hence $e = 1$, $f = 3$, $p = 0$ and $r = 3$. 
**Definition.** We denote by $\tau$ the substitution that has

$$M_\tau := \begin{pmatrix} c + d + dg & a + b + bg - (c + d + dg) \\ c + dg & a + bg - (c + dg) \end{pmatrix}$$

as incidence matrix, and which is such that

- if we cyclically shift $\tau(0)$ over $r$ positions to the right, we get a Christoffel word,

- the $(r + 1)$th letter of $\tau(0)$ is underlined,

- $\tau(1)$ equals the left $a + bg$ letters of $\tau(0)$.

**Example 5 (Continued).** We had $M_\sigma = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$, $\sigma(0) = 01001$, $\sigma(1) = 010010101001$ and $r = 3$.

We get $M_\tau = \begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$ and $\tau(0) = 011010101$, $\tau(1) = 0110101$. 
**Theorem.** Let $\sigma$ be a Sturmian substitution that has an incidence matrix with determinant 1, that has a fixed point starting with 0, and that is not a Christoffel substitution. Let the Sturmian substitution $\tau$ and the central word $v_n$ for $n \geq 1$ be defined as before. Then $\tau(v_n) = v_{n+1}$.

**Remark.** If the substitution $\tau$ has a fixed point starting with 0, we can apply the procedure again, and call the result $\phi$. If $g = 0$ then $M_\phi = M_\sigma$, hence $\phi(0), \phi(1)$ are cyclic shifts of $\sigma(0), \sigma(1)$ respectively. In case $\sigma$ is a Christoffel substitution, we get $\phi = \sigma$. This is not true in general.
Example 6. Let $M_{\sigma} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ and $\sigma(0) = 0101101$, $\sigma(1) = 01101$. Then $e = 1$, $f = 1$, $p = 0$ and $r = 5$. We get $M_{\tau} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ and $\tau(0) = 0100100$, $\tau(1) = 010$. Repeating this process, we get $e = 4$, $f = 1$, $p = 1$ and $r = 4$, which results in $\phi(0) = 1011010$, $\phi(1) = 10110$. 
What if determinant is -1?

We can still form the central words $v_n$, except that for odd $n$, we need to reflect the central progressions $w_n$ in the origin, before we construct $v_n$ from $w_n$.

Since the substitution $\sigma^2$ has incidence matrix with determinant 1, it is clear that there exists a substitution $\tau_2$ such that $v_{2n} = \tau_2(v_{2n-2})$. But as the following example shows, there does not need to exist a substitution $\tau$ such that $v_n = \tau(v_{n-1})$. 
Example 7. Let $M_{\sigma} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma(0) = 001$, $\sigma(1) = 0$. Then we get the following table for $v_n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$v_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 1 0</td>
</tr>
<tr>
<td>2</td>
<td>1 1 0 1 0 1 0 1 0</td>
</tr>
<tr>
<td>3</td>
<td>1 1 0 1 0 1 1 0 1 0 1 0</td>
</tr>
</tbody>
</table>

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It is easy to check that there is no substitution $\tau$ such that $\tau(v_2) = v_3$.

However, this example suggests that if we define $\tau$ by $\tau(0) = 110$, $\tau(1) = 10$, and $\tau(ab) = \tau(b)\tau(a)$ for every $a, b$ in $A$, then we have $v_n = \tau(v_{n-1})$.

An interesting question is if similar functions exist for all Sturmian substitutions that have an incidence matrix with determinant $-1$. 