Topological properties of central tiles for substitutions

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Mars 2007
Central Tiles and Rauzy fractals

Introduced by Rauzy and Thurston in different frameworks

- **Symbolic dynamical systems** Geometrical representation of the shift map on a substitutive dynamical system. The shift map commutes with a piecewise exchange of domains.

- **Beta-numeration** Geometric compact representation of real numbers with an empty fractional greedy expansion in a non-integer numeration system.

- **Discrete geometry** Renormalized limit of an inflation action on faces of discrete planes.
Specific topological properties

0 inner point

connectivity

Haussdorf dimension of the boundary

disklikeness
Parametrization of the boundary

(0 inner point)

(not connected)

Give criterions for topological properties that can be checked algorithmically?
Definitions

- **Substitution.** Endomorphism $\sigma$ of the free monoid $\{0, \ldots, n\}^*$.
  
  \[
  \sigma : \quad 1 \rightarrow 12 \quad 2 \rightarrow 13 \quad 3 \rightarrow 1. \quad (\beta^3 = \beta^2 + \beta + 1)
  \]

- **Primitivity.** The map $M$ obtained by abelianization of $0, \ldots, n^*$ on $\sigma$ is primitive.

- **Periodic points.** If $\sigma$ is primitive, then there exists at least a periodic point $w$ for $\sigma$:
  
  \[
  \sigma^\nu(w) = w.
  \]

- **unit Pisot assumption** The dominant eigenvalue $\beta$ of the abelianized matrix of $\sigma$ is a unit Pisot number.

  \[
  \sigma : \quad 1 \rightarrow 12 \quad 2 \rightarrow 3 \quad 3 \rightarrow 1 \quad 4 \rightarrow 5 \quad 5 \rightarrow 1 \quad (\beta^3 = \beta + 1)
  \]

  Let $d \leq n$ be the algebraic degree of $\beta$. Let $\text{Min}_\beta$ be its minimal polynomial.
Central Tile

- **Beta-decomposition of the space:**
  - **Beta-expanding line** \( H_e \)
  - **Beta-contracting space** \( H_c \) generated by the eigenvectors for the algebraic conjugates \( \beta_i \)'s of \( \beta \).
  - **Beta-Orthogonal space:** subspace \( H_o \) generated by the other eigenvectors.

- **Beta-projection:** projection on the beta-contracting plane parallel to \( GH_e + H_o \)
  \[
  \forall w \in A^*, \pi(l(\sigma(w))) = h\pi(l(w)).
  \]
Construction of the central tile

- Compute a periodic point
- Embed it as a stair in $\mathbb{R}^n$.
- Project the stair on the beta-contracting plane
- Keep memory of the type of step when projecting
- Take the closure

$\sigma(1) = 112$, $\sigma(2) = 113$, $\sigma(3) = 4$, $\sigma(4) = 1$

112 112 113 112 112 113 112 112 4 112 112
113 112 112 113 112 112 113 112 112 4 112
112 113 112 112 113 1 112 ...
Central Tile

Construction of the central tile

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Definition

Let $\sigma$ be a primitive unit Pisot substitution. The central tile of $\sigma$ is defined by

$$\mathcal{T}_\sigma = \overline{\{\pi(l(u_0 \cdots u_{i-1})); \ i \in \mathbb{N}\}}.$$ 

Subtile: $\mathcal{T}(a) = \overline{\{\pi(l(u_0 \cdots u_{i-1})); \ i \in \mathbb{N}, \ u_i = a\}}$. 
Main topological properties

**Theorem**

Let \( \sigma \) be a primitive Pisot unit substitution.

- The central tile \( T \) is a **compact** subset of \( \mathbb{R}^{d-1} \), with **nonempty interior** and **non-zero measure**. (\( d \) degree of \( \text{Min}_\beta \)).
- Each subtile is the **closure** of its **interior**.
- The subtiles of \( T \) are solutions of the following affine **Graph Iterated Function System**:
  \[
  T(a) = \bigcup_{b \in A, \sigma(b) = \text{pas}} h(T(b)) + \pi(l(p))
  \]
- The subtiles are **disjoint** when the substitution satisfies the so-called **coincidence condition**.

\[
\begin{align*}
T(1) &= h[T(1) \cup (T(1) + \pi l(e_1))] \\
&\quad \cup T(2) \cup (T(2) + \pi l(e_1)) \cup T(4), \\
T(2) &= h(T(1) + 2\pi l(e_1)), \\
T(3) &= h(T(2) + 2\pi l(e_1)), \\
T(4) &= h(T(3))
\end{align*}
\]

\( \sigma(1) = 112, \sigma(2) = 113, \sigma(3) = 4, \sigma(4) = 1 \)
Specific topological properties

- **Inner point**: (Sufficient conditions, CNS conditions) [Rauzy, Akiyama]
- **Connectivity**: (Sufficient condition, necessary condition) [Canterini, Messaoudi]
- **Haussdorff dimension of the boundary**: (Examples of computation) [Feng-Furukado-Ito, Thuswaldner]
- **Disklikeness**: Parametrization of the boundary (Examples) [Messaoudi, Sirvent]

Give criterions for topological properties that can be checked algorithmically?
The main object: tilings

A **multiple tiling** is given by a *translation set* \( \Gamma \subset H_c \times A \) such that

- \( H_c = \bigcup_{(\gamma, i) \in \Gamma} T_i + \gamma \)

- Delaunay set (finite number of intersections for a given tile).
- almost all points in \( H_c \) are covered exactly \( p \) times.

**Self-replicating substitution multiple tiling**

\[
\Gamma_{srs} = \left\{ (\pi(x), i) \in \pi(\mathbb{Z}^n) \times A, \right.
\]
\[
0 \leq \langle x, v_\beta \rangle < \langle e_i, v_\beta \rangle \right\}.
\]

Delaunay set, self-replicating, aperiodic and repetitive.

Tiling iff super-coincidence.
The main object: tilings

A multiple tiling is given by a \textit{translation set} $\Gamma \subset \mathbb{H}_c \times \mathcal{A}$ such that

- $\mathbb{H}_c = \bigcup_{(\gamma, i) \in \Gamma} T_i + \gamma$
- Delaunay set (finite number of intersections for a given tile).
- almost all points in $\mathbb{H}_c$ are covered exactly $p$ times.

\textbf{Self-replicating substitution multiple tiling}

$$\Gamma_{srs} = \{(\pi(x), i) \in \pi(\mathbb{Z}^n) \times \mathcal{A}, 0 \leq \langle x, v_\beta \rangle < \langle e_i, v_\beta \rangle\}.$$  

\textbf{Lattice multiple tiling}

$$\Gamma_{lattice} = \{(\pi(x), i) \in \pi(\mathbb{Z}^n) \times \mathcal{A}, \sum_{1}^{d} \langle x, e_{B(k)} \rangle = 0\}.$$  

Delaunay set, self-replicating, aperiodic and repetitive.

Tiling iff super-coincidence.

Periodic and Delaunay set.
When $\sigma$ irreducible, tiling iff super-coincidence.
The main tool: IFS description of intersection of tiles

Suppose that two tiles intersect. \( I = \mathcal{I}(a) \cap (\pi(x) + \mathcal{I}(b)) \neq \emptyset \).

Each tile admits a decomposition, hence

\[
\mathcal{I}(a) = \bigcup_{\sigma(a_1) = p_1 a s_1} h(\mathcal{I}(a_1) + \pi l(p_1)), \quad \mathcal{I}(b) = \bigcup_{\sigma(b_1) = p_2 b s_2} h(\mathcal{I}(b_1) + \pi l(p_2)).
\]

Then the union can be rewritten as

\[
I = \bigcup_{\sigma(a_1) = p_1 a s_1, \sigma(b_1) = p_2 b s_2} h[I(a_1) + \pi l(p_1)] \cap \{h[I(b_1) + \pi l(p_2)] + \pi(x)\}.
\]

\[
= \bigcup h\pi_l(p_1) + h[I(a_1) \cap (\mathcal{I}(b_1) + \pi l(p_2) - \pi l(p_1) + h^{-1}\pi(x))]
\]

The **boundary graph** maps the intersection of two tiles to each intersections that is contained in it (up to a translation).

\[
(0, a) \cap (\pi(x), b) \rightarrow (0, a_1) \cap (\pi l(p_2) - \pi l(p_1) + h^{-1}\pi(x), b_1)
\]
The main tool: IFS description of intersection of tiles

Suppose that two tiles intersect. \( \mathcal{I} = \mathcal{T}(a) \cap (\pi(x) + \mathcal{T}(b)) \neq \emptyset \).
Each tile admits a decomposition, hence

\[
\mathcal{T}(a) = \bigcup_{\sigma(a_1) = p_1 \alpha_1} h(\mathcal{T}(a_1) + \pi l(p_1)).
\]
\[
\mathcal{T}(b) = \bigcup_{\sigma(b_1) = p_2 \beta_2} h(\mathcal{T}(b_1) + \pi l(p_2)).
\]

Then the union can be rewritten as

\[
\mathcal{I} = \bigcup_{\sigma(a_1) = p_1 \alpha_1, \sigma(b_1) = p_2 \beta_2} h[\mathcal{T}(a_1) + \pi l(p_1)] \cap \{ h[\mathcal{T}(b_1) + \pi l(p_2)] + \pi(x) \}.
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\]
The main tool: IFS description of intersection of tiles

Suppose that two tiles intersect. \( I = \mathcal{T}(a) \cap (\pi(x) + \mathcal{T}(b)) \neq \emptyset \).
Each tile admits a decomposition, hence

\[
\begin{align*}
\mathcal{T}(a) &= \bigcup_{\sigma(a_1) = p_1 a s_1} h(T(a_1) + \pi l(p_1)). \\
\mathcal{T}(b) &= \bigcup_{\sigma(b_1) = p_2 b s_2} h(T(b_1) + \pi l(p_2)).
\end{align*}
\]

Then the union can be rewritten as

\[
I = \bigcup_{\sigma(a_1) = p_1 a s_1, \sigma(b_1) = p_2 b s_2} h[T(a_1) + \pi l(p_1)] \cap \{ h[T(b_1) + \pi l(p_2)] + \pi(x) \}.
\]

The boundary graph maps the intersection of two tiles to each intersections that is contained in it (up to a translation).

\((0, a) \cap (\pi(x), b) \rightarrow (0, a_1) \cap (\pi l(p_2) - \pi l(p_1) + h^{-1}\pi(x), b_1)\)
Self-replicating substitution neighbor graph

- **Nodes:** pairs of faces $[(0, a), (\pi(x), b)]$ such that
  - $(\pi(x), b) \in \Gamma_{srs}$ (points in the translation set)
  - $||\pi(x)|| \leq ||T||$ (if not, the intersection is empty)
- There is an edge between $(0, a) \cap (\pi(x), b)$ and $(0, a_1) \cap (\pi(x_1), b_1)$ if $T(a_1) \cap (\pi(x) + T(b_1))$ appears up to a translation in the decomposition of $T(a) \cap (\pi(x) + T(b))$.

**Theorem**

The self-replicating substitution boundary graph is finite.

$T(a) \cap (\pi(x) + T(b))$ is nonempty iff the self-replicating substitution boundary graph contains an infinite walk starting in $[(0, a), (\pi(x), b)]$.

Each path of the graph correspond to a point lying at the intersection. The boundary graph is a GIFS description of the boundary of the central tile.
Self-replicating substitution neighbor graph

- **Nodes:** pairs of faces $[(0, a), (\pi(x), b)]$ such that
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- There is an edge between $(0, a) \cap (\pi(x), b)$ and $(0, a_1) \cap (\pi(x_1), b_1)$ if $T(a_1) \cap (\pi(x) + T(b_1))$ appears up to a translation in the decomposition of $T(a) \cap (\pi(x) + T(b))$.

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Each path of the graph correspond to a point lying at the intersection. The boundary graph is a GIFS description of the boundary of the central tile.
Example

The central subtiles intersect 17 other tiles in the SRS tiling.

$T(1)$ has 5 neighbours outside the central tile.
Several graphs

It is algorithmically possible to compute graphs

- **Self-replicating substitution neighbor graph** Pairs of tiles intersecting in the SRS multiple tiling.
- **Connectivity graph** Pairs of subtiles of \( T(a) \) with a common point.
- **Lattice neighbor graph** Pairs of tiles in lattice multiple tiling.
- **Triple point neighbor graph** Triplets of tiles intersecting in the SRS multiple tiling.
- **Quadruple point neighbor graph** Quadruplets of tiles intersecting in the SRS multiple tiling.

- 6 intersecting pairs in the lattice tiling
- 20 intersecting triplets in the SRS tiling (redundancy)
- 4 intersecting quadruplets in the SRS tiling
Application to boundary

**Proposition**

The SRS multiple tiling is a tiling iff the dominant eigenvalue of the matrix of the SRS neighbor graph is strictly less than $\beta$.

The lattice multiple tiling is a tiling iff the dominant eigenvalue of the matrix of the lattice neighbor graph is strictly less than $\beta$.

Application: $\sigma(1) = 112, \sigma(2) = 113, \sigma(3) = 4, \sigma(4) = 1$ generates a lattice tiling.

$\sigma(1) = 12, \sigma(2) = 13, \sigma(3) = 4, \sigma(4) = 5, \sigma(5) = 1$ does not generate a lattice tiling with the given vectors.
Application to boundary

Proposition
Let $\lambda$ be the largest conjugate of $\beta$ and $\lambda'$ the smallest conjugate. Let $\mu$ be the dominant eigenvalue of the matrix of the SRS neighbor graph. If the SRS neighbor graph is strongly connected then

$$\dim_B(\partial T) = \dim_B(\partial T(a)) = d - 1 + \frac{\log \lambda - \log \mu}{\log \lambda'}$$

Application: Explicit computations of Haussdorff dimensions.
Application to connectivity

Connectivity graph For each subtile $T(a)$, there is an edge between two subunits iff they intersect.

Proposition

Each $T(a)$ is a locally connected continuum if and only if the connectivity graph $G_a(V, E)$ is connected for each $a \in A$. $T$ is connected iff each $T(a)$ and the subtiles have connections.

$\sigma(1) = 3; \sigma(2) = 23, \sigma(3) = 31223.$

- The three central tiles intersect.
- One subtile of $T(2)$ intersects no other subtile: some nodes are missing in the graph.
**Criterion for non disklike**

**Proposition**

Suppose that $\beta$ has degree 3. If the central tile $T$ is homeomorphic to a closed disk then $T$ has at most six neighbors $\lambda$ in a lattice tiling with the property

$$|T_\sigma \cap (T_\sigma + \gamma)| > 1.$$ 

Deduced from Bandt and Gelbrich.

**Application:** when there is lattice tiling, check if the central tile is not disklike.

8 neighbours. Not homeomorphic to a closed disk.  

Only 6 neighbors. No conclusion.
Theorem

Suppose that $\beta$ has degree 3. Let $B_1, \ldots, B_k$ be the boundary pieces $T(a) \cap (T(b) + \pi(x))$. Suppose that

- The $B_i$'s form a circular chain: they can be arranged so that they have one intersection point with the following and no intersection with the others.
- The self-affine decomposition of each $B_i$ is a regular chain

Then the central tile is disklike

Translation into the boundary graph framework. A boundary piece $B_i$ corresponds to a node $[(0, a), (\pi x, b)]$. 

Criterion for disklike
Algorithmic criterion for disklike

- Identify pairs intersecting as a singleton
- Check that every triple intersection in a singleton.
- For every pair-intersection \([(0, a), (\pi x, b)]\) that is not a singleton, check that it intersect exactly two other intersections.
- The intersections make a loop.
- Similar checking for the successors of \([(0, a), (\pi x, b)]\).

\[
\sigma(1) = 112, \ \sigma(2) = 113, \ \sigma(3) = 4, \ \sigma(4) = 1
\]

- 17 pair-intersections of tiles.
- 4 contains exactly one point (Sommets 1, 15, 16, 17)
- 13 remaining infinite pair-intersections.

The central tile for \(\sigma(1) = 112, \ \sigma(2) = 113, \ \sigma(3) = 4, \ \sigma(4) = 1\) is homeomorphic to a closed disk.
Criterion for not simply connected

Theorem

*The SRS boundary graph, triple point graph and quadruple point graph allow to check a condition for not simply connected.*
Conclusion

- Many topological properties of central tiles can be checked.
- Understand the structure of boundary, triple and quadruple graphs for classes of substitutions to deduce general properties?
- What is the relation between topological properties and ergodic properties of the substitutive dynamical system?
- What can be deduced from topological properties about beta-numeration systems?
- (Find a good programmer to compute efficiently the graphs to check the conditions?)