The Normal Law, Free Probability and a Hopf Algebra of Rooted Binary Trees

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Free Probability has been introduced by Voiculescu to study the harmonic analysis of the free group and its von Neumann algebra. In this theory the notion of independence of random variables is replaced by *freeness* of noncommutative random variables, which are elements of noncommutative von Neumann algebras. Free probability shares many featurs with classical probabilities, e.g., *free convolution* of measures is obtained by adding free random variables just like classical convolution of measures on the real line is connected to addition of independent random variables. There is a central limit theorem featuring the Wigner semicircle law, there are a free Brownian motion, free infinitely divisible distributions, etc. In fact there is the so-called *Bercovici-Pata bijection* between classical infinitely divisible distributions and the free infinitely divisible distributions which maps the respective domains of attraction onto each other. Examples of classical infinite divisible distributions are the normal law, the Poisson law and the stable laws in classical probability. Their freely infinitely divisible counterparts are the Wigner law, the Marchenko-Pastur distribution and the free stable laws.

In the present work we prove the somewhat strange fact that the classical normal distribution is also freely infinitely divisible. This is quite unexpected as there are only two other nontrivial examples measures with the property of being infinitely divisible both in the classical and the free sense.

The proof amounts to showing that the sequence of the so-called *free cumulants* of the normal distribution is a moment sequence. This sequence counts the number of connected set partitions and has been studied by Touchard, Riordan and others previously. We review a few other combinatorial interpretations, from computer science to Hopf algebras of rooted binary trees which appeared in the context of renormalization theory.