THU/EPCOS
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A mean-square bound for the lattice discrepancy of a torus in $\mathbb{R}^3$

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For any compact body $\mathscr{B}$ in $\mathbb{R}^3$ and a large real variable t, the *lattice point discrepancy* of the linearly dilated body $t\mathscr{B}$ is defined by

$$P_{\mathscr{B}}(t) := \#(\mathbb{Z}^3 \cap t\mathscr{B}) - \mathrm{vol}(\mathscr{B})t^3 .$$

This talk will concentrate on *mean-square estimations* concerning this quantity. In fact, for $\mathscr{B} = \mathscr{B}_0$ the unit ball in $\mathbb{R}^3$, it has been proved by V. Jarnik in 1940 that, for some positive constant c,

$$\int_0^T (P_{\mathscr{B}_0}(t))^2 \mathrm{d}t \sim c\,T^3 \log T .$$

For a general convex body $\mathscr{B}$ in $\mathbb{R}^3$ with smooth boundary of nonvanishing Gaussian curvature, A. Iosevich, E. Sawyer, and A. Seeger [2] recently established the comparable upper bound $O(T^3(\log T)^2)$.

The generic example of a non-convex body bounded by a smooth surface of genus 1 is the torus $\mathscr{T} = \mathscr{T}_{a,b}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (a+r\cos\alpha)\cos\beta \\ (a+r\cos\alpha)\sin\beta \\ r\sin\alpha \end{pmatrix}, \qquad 0 \le \alpha,\beta < 2\pi, \quad 0 \le r \le b,$$

with fixed constants $a > b > 0$. It is known that

$$P_{\mathscr{T}}(t) = \mathscr{F}_{a,b}(t)\,t^{3/2} + \Delta_{\mathscr{T}}(t) .$$

Here $\mathscr{F}_{a,b}(t)$ is an explicit continuous periodic function, while $\Delta_{\mathscr{T}}(t) \ll t^{\frac{3}{2}-\frac{1}{286}+\varepsilon}$ according to D. Popov [4], and $\Delta_{\mathscr{T}}(t) \ll t^{\frac{3}{2}-\frac{1}{8}+\varepsilon}$ according to W.G. Nowak [3]. The main term $\mathscr{F}_{a,b}(t)t^{3/2}$ comes from the infinitely many points of Gaussian curvature zero on the boundary.

The goal of this talk is to sketch the argument leading to the (possibly expected) mean-square bound

$$\int_0^T (\Delta_{\mathscr{T}}(t))^2 \mathrm{d}t \ll T^{3+\varepsilon} .$$

[1] V.C. Garcia and W.G. Nowak, *A mean-square bound concerning the lattice discrepancy of a torus in* $\mathbb{R}^3$, manuscript.

[2] A. Iosevich, E. Sawyer, and A. Seeger, *Mean square discrepancy bounds for the number of lattice points in large convex bodies*, J. Anal. Math. **87** (2002), 209-230.

[3] W.G. Nowak, *The lattice point discrepancy of a torus in* $\mathbb{R}^3$, Acta Math. Hung. **120** (2008), 179-192.

[4] Popov D. Popov, *On the number of lattice points in three-dimensional bodies of revolution*, Izv. Math. **64** (2000), 343-361. Translation from Izv. RAN, Ser. Mat. **64** (2000), 121-140.