Optimal Error Estimates in Stochastic Homogenization

Mon/P1
09:30–10:20

Felix Otto (Univ. Bonn)

In applications like water flow through a porous medium, one has to solve an elliptic equation $-\nabla\cdot(a(x)\nabla u)=f$ with heterogeneous coefficients $a(x)$ that vary on a characteristic length scale ℓ much smaller than the domain size. The coefficients $a(x)$ are typically characterized in statistical terms: Their statistics are assumed to be stationary (i. e. translation invariant) and to decorrelate over distances large compared to ℓ. In this situation, it is known from the theory of stochastic homogenization that the solution operator, i. e. the inverse of the elliptic operator $-\nabla\cdot a(x)\nabla$, behaves – on scales large compared to ℓ – like the inverse of $-\nabla\cdot a_{hom}\nabla$ with *homogeneous, deterministic* coefficients a_{hom}. This is a major reduction in complexity.

Hence the relation between the statistics of the heterogeneous coefficients $a(x)$ and the value of the homogenized coefficient a_{hom} is of practical importance. Stochastic homogenization also provides a formula for a_{hom} in terms of the ensemble of $a(x)$. It involves the solution of the “corrector problem” $-\nabla\cdot(a(x)(\nabla\phi_\xi+\xi))=0$ in the whole space $\mathbb{R}^d$ for a given direction ξ. The formula then is given by $a_{hom}\xi=\langle a(\nabla\phi_\xi+\xi)\rangle$, where $\langle\cdot\rangle$ denotes the ensemble average. Despite its simplicity, this formula has to be approximated in practise:

1) The corrector problem can only be solved for a small number of realizations of the coefficients $a(x)$. Thus, appealing to ergodicity, the ensemble average has to be replaced by a spatial average over a region of large diameter L.
2) The corrector problem can only be solved in a finite domain of large diameter L, thereby introducing some artificial boundary conditions.

In this talk, we present optimal estimates on both errors for the simplest possible model problem: We consider a discrete elliptic equation on the d-dimensional lattice $\mathbb{Z}^d$ with random coefficients a, which are identically distributed and independent from edge to edge. This makes a connection with the area of “random walks in random environments”. We establish the optimal scaling of both errors in the ratio $L/\ell\gg 1$ (where the correlation length ℓ is unity in our model problem). It turns out that the scaling is the same as in the case of coefficients $a(x)$ that are very close to the identity (where the corrector problem can be linearized). Hence with respect to the error scaling, the highly nonlinear relation between $a(x)$ and a_{hom} behaves like the linearized one.

Our methods involve spectral gap estimates and estimates on the Green’s function which only depend on the ellipticity ratio λ of a.

This is joint work with Antoine Gloria, INRIA Lille.