Rates in Local Limit Theorems for Sums of Partial Quotients of Continued Fractions

TUE/EPCOS
11:00–11:20

Lothar Heinrich (Univ. Augsburg)

We consider the continued fraction expansion $a_1(\omega) = [1/\omega]$ and $a_k(\omega) = a_1(T^{k-1}\omega)$ for $k \geq 2$ (with the 'continued fraction mapping' $T\omega = 1/\omega - [1/\omega]$) of a real number $\omega \in (0,1)$ drawn according to some probability measure P having a strictly positive, Lipschitz-continuous Lebesgue density $p(\cdot)$. The most important examples for such P are the uniform distribution with $p(\omega) \equiv 1$ or the Gauss-Kuzmin-Lévy distribution with $p(\omega) = 1/(1+\omega)\log 2$. In our talk we are mainly interested in the local asymptotic behaviour of the sum $S_n = a_1 + \cdots + a_n$, that is, in the approximation of the point probabilities $\mathsf{P}(S_n = m)$ for $m \geq n$. It is well-known that the distribution function $\mathsf{P}(S_n/n \leq A_n + x)$ with $A_n = (\log n - \gamma)/\log 2$ ($\gamma =$ Euler-Mascheroni constant) converges uniformly on the whole axis to a stable distribution function $G(x)$ with characteristic exponent $\alpha = 1$ and skewness parameter $\beta = 1$, see [1], [2]. Even a uniform rate $\mathsf{O}(\log^2 n/n)$ of the sup-distance could be obtained in [1]. In the present paper we prove the corresponding local limit theorem

$$\sup_{m \geq n} \Big| \mathsf{P}\big(S_n = m\big) - \frac{1}{n} g\Big(\frac{m}{n} - A_n\Big) \Big| = \mathsf{O}\Big(\frac{\log^2 n}{n^2}\Big) \quad \text{as} \quad n \longrightarrow \infty \tag{1}$$

with the stable probability density function $g(x) = G'(x)$. The crucial difficulty in proving (1) consists in the need of sufficiently small bounds of the Fourier transform $f_n(t) = \int_0^1 \exp\{itS_n(\omega)\} p(\omega)\mathrm{d}\omega$ uniformly in $[\varepsilon, 2\pi - \varepsilon]$ for any $\varepsilon > 0$. The algebraic properties and the 'almost-Markov property' of the sequence $a_1, a_2, \ldots$ yield the uniform estimate $|f_n(t)| = \mathsf{O}\big(\exp\{-b(\varepsilon)\sqrt{n}\}\big)$ with some constant $b(\varepsilon) > 0$. This combined with the techniques developed in [1] leads to (1).
Finally, we give reasons that the obtained rates of convergence in local as well as in the integral limit theorem in [1] cannot be improved.

[1] L. Heinrich: Rates of convergence in stable limit theorems for sums of exponentially $\psi-$mixing random variables with an application to the metric theory of continued fractions. *Math. Nachr.* **181** (1987), 185–214.
[2] D. Hensley: The statistics of the continued fraction digit sum. *Pacific J. Math.* **192** (2000), 103–120.