On integer conjugacy classes of SL(3,$\mathbb{Z}$)

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FRI/EPCOS
16:30–16:50

In this talk we discuss the classical problem of description of conjugacy classes in the group $\mathrm{SL}(n,\mathbb{Z})$. The conjugacy problem is solved only for $n = 2$ and there is no any clear description of conjugacy classes for the case $n \geq 3$. In this talk we make first steps to understand the structure of the set of conjugacy classes in $SL(n,\mathbb{Z})$. Our main results are in the three-dimensional case, nevertheless, the observed phenomena seems to be common in the higher dimensions.

Unlike the case of complex matrices, the conjugacy classes of $\mathrm{SL}(n,\mathbb{Z})$-matrices with all distinct eigenvalues are not uniquely determined by the coefficients of characteristic polynomials. Therefore, it is not possible to enumerate such classes by Jordan canonical forms. We propose to replace Jordan forms by reduced perfect Hessenberg matrices. Still these matrices do not give the complete invariant of conjugacy classes, nevertheless the study of $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{SL}(3,\mathbb{Z})$ shows that the set of Hessenberg matrices is a “nice approximation” to the set of conjugacy classes. For instance, for the set of $\mathrm{SL}(3,\mathbb{Z})$-matrices with irreducible characteristic polynomial over rational numbers having one real and two complex roots we obtain the following: perfect Hessenberg matrices distinguish conjugacy classes asymptotically.

The solution to the problem in the two-dimensional case is a subject of Gauss Reduction Theory (see, for instance in [3]). The idea of this theory is to find special reduced matrices in each conjugacy class. The number of such matrices in a conjugacy class equals to the length of a minimal even period of ordinary continued fraction associated to the conjugacy class. We put together old constrictions of Klein’s polyhedron [2] and Voronoi’s continued fractions [4] and the ideas of J. A. Buchmann [1] to obtain the general definition of geometric Klein-Voronoi multidimensional continued fractions. In our study of conjugacy questions we work with periods of Klein-Voronoi continued fractions.

[1] J. A. Buchmann: A generalization of Voronoi’s algorithm I, II, *Journal of Number Theory*, **20**(1985), pp. 177–209.

[2] F. Klein: Ueber einegeometrische Auffassung der gewöhnliche Kettenbruchentwicklung, *Nachr. Ges. Wiss. Göttingen Math-Phys. Kl.*, **3** (1895), pp. 357–359.

[3] J. Lewis, D. Zagier: Period functions and the Selberg zeta function for the modular group, *in The Mathematical Beauty of Physics, Adv. Series in Math. Physics 24, World Scientific*, Singapore (1997), pp. 83–97.

[4] G. F. Voronoy: On a Generalization of the Algorithm of Continued Fractions, *Izd. Varsh. Univ., Varshava* (1896); *Collected Works in 3 Volumes, Izd. Akad. Nauk Ukr*, SSSR, Kiev (in Russian), vol. 1, 1952.