THU/EPCOS
11:30–11:50

Additive functions on polynomial sequences

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Let $q \geq 2$ be an integer. Then we can represent every positive integer n in a unique way as follows

$$n = \sum_{r\geq 0} d_r(n) q^r \quad \text{with} \quad d_r(n) \in \{0,\dots,q-1\}.$$

We call this the *q-ary representation* of n with q the *base* and $\{0,\dots,q-1\}$ the *set of digits*.

If a function f acts only on the digits of a representation and is independent of the position of the digit, we call it *strictly q-additive*, *i.e.*,

$$f(n) = \sum_{r\geq 0} f\left(d_r(n)\right),$$

where n is as above.

A simple example of such a function is the sum-of-digits function s_q which is defined by

$$s_q(n) = \sum_{r\geq 0} d_r(n),$$

where n again is as above. In the present talk we want to concentrate on the summatory function of f. For the case of the sum-of-digits function Delange [1] was able to show

$$\sum_{n\leq N} s_q(n) = \frac{q-1}{2} N \log_q N + NF\left(\log_q N\right),$$

where $\log_q$ is the logarithm to base q and F a 1-periodic, continuous and nowhere differentiable function.

In the present talk we want to extend this result on the one hand to arbitrary strict q-additive functions f and on the other hand to polynomial sequences. Let μ_f be the *mean* of the values of f, *i.e.*,

$$\mu_f := \frac{1}{q} \sum_{a=0}^{q-1} f(a)$$

Then our main result reads as follows.

THEOREM. *Let $q \geq 2$ be an integer, f be a strictly q-additive function and p be a polynomial of degree $k \geq 1$. Then there are $c \in \mathbb{R}$ and $\varepsilon > 0$ such that*

$$\sum_{n\leq N} f\left(\lfloor p(n) \rfloor\right) = \mu_f N \log_q(p(N)) + cN + NF\left(\log_q(p(N))\right) + \mathcal{O}\left(\frac{N}{\log N}\right),$$

where $\lfloor x \rfloor$ is the greatest integer less than x and F is a 1-periodic function depending on q, f and p.

[1] H. DELANGE: Sur la fonction sommatoire de la fonction“somme des chiffres”, *Enseignement Math. (2)* **21** (1975), no. 1, 31–47.