Optimal Transport and Geometry

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The theory of optimal transports, originating in the classical problem of mass transportation due to Monge (1781) and its weak reformulation by Kantorovich (1942), has undertaken an impressive development in the last decade, initiated by fundamental works of Brenier, McCann, Otto, Villani and others. One of the remarkable observations is that applying geometric concepts to the space of probability measures – the so-called Wasserstein space – allows to derive surprising new results and, vice versa, properties of the Wasserstein space can be used to analyze the geometry of the underlying space.

In the talk, we will first give a brief introduction to the theory of optimal transports. In particular, we will present the Riemannian structure on the space of probability measures induced by the optimal transports on Euclidean space or on a given Riemannian manifold. This picture, for instance, allows to identify the heat equation as the gradient flow for the relative entropy. Indeed, this identification holds true in great generality including Finsler spaces, sub-Riemannian geometries like the Heisenberg group and infinite dimensional spaces like the Wiener space.

Then we present the concept of generalized lower Ricci curvature bounds for metric measure spaces (M,d,m), introduced by Lott & Villani and the author. These curvature bounds are defined in terms of optimal transports, more precisely, in terms of convexity properties of the relative entropy regarded as function on the Wasserstein on the given space M. For Riemannian manifolds, this turns out to characterize lower Ricci bounds. Other important examples covered by this concept are Alexandrov spaces and Finsler manifolds.

One of the main results is that these generalized lower curvature bounds are stable under (e.g. measured Gromov-Hausdorff) convergence. In combination with an upper bound on the dimension, they imply sharp versions of the Brunn-Minkowski inequality, of the Bishop-Gromov volume comparison theorem and of the Bonnet-Myers theorem.

Finally, we explain how convexity properties of the relative entropy on the Wasserstein space are related with functional inequalities (e.g. logarithmic Sobolev inequalities), with concentration of measure and with contraction properties of the evolution semigroups. We also sketch some recent applications and links to Ricci flow which allow to derive or reformulate various monotonicity formulas of Perelman in terms of optimal transports.