A second look at binary digits

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When saying that we expressed a natural number using “binary digits”, of course we rather mean that we used the binary positional number system, which consists of the base, which is 2, and the digits, which are 0 and 1. In this talk, I will consider a few variations on this theme.

First, in the usual binary system (and in the usual decimal system as well), digits are not enough; we need a minus sign to be able to express also negative numbers in our system. I will show that for the system with base 2, this is necessary: we cannot choose two digits so that all integers correspond to a finite sequence of digits alone.

This changes when we take -2 as a base instead of 2: it is easy to show that all integers do have such an expansion with this base, still using digits 0 and 1.

My own variation on this topic is the following question: can we characterise *all sets of two digits* that permit to express all integers, on base -2? It will turn out that the answer is Yes, and that there is an infinitude of “good” digit sets for base -2, which has a surprising and elegant structure [3, Theorem 4.1].

Finally, we will get into some deeper water and discuss ongoing work on generalisations of this result, where the base is no longer ± 2 itself, but some algebraic integer α that has *norm* ± 2 and moreover has the property that all its conjugates are greater than 1 in absolute value. Here we encounter interesting problems of algebraic number theory and fractal geometry [1,2].

[1] SHIGEKI AKIYAMA and JÖRG M. THUSWALDNER: A survey on topological properties of tiles related to number systems, *Geometriae Dedicata* **109** (2004), 89–105.

[2] JEFFREY C. LAGARIAS and YANG WANG: Haar bases for $L^2(\mathbf{R}^n)$ and algebraic number theory, *J. Number Theory* **57** (1996), 181–197. Addendum/corrigendum: *J. Number Theory* **76** (1999), 330–336.

[3] CHRISTIAAN E. VAN DE WOESTIJNE: Noncanonical number systems in the integers. *J. Number Theory* **128** (2008), 2914–2938.