The discrete Gross-Pitaevskii model and condensation in the single particle ground state

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Definition of the discrete Gross-Pitaevskii model

Definition (Discrete Gross-Pitaevskii model)

Denote by

- $\Lambda_L = [-L, L]^d$ the cube of side length $L > 0$ and volume $|\Lambda_L| = (2L)^d$,
- $\mathrm{N} = \varrho|\Lambda_L|$, $\varrho > 0$ the number of particles,
- $\mathrm{H}^N_{\omega,L} = (-\Delta + V_\omega)^N_{\Lambda_L}$ the discrete Schrödinger operator on Λ_L with Neumann boundary conditions. Here $(-\Delta)^N_{\Lambda_L}$ is the discrete Laplacian and concerning the i.i.d. random potential $V_\omega \geq 0$ we assume a regular distribution,
- $\mathrm{UN}\|\varphi\|_4^4$ is the interaction energy. Concerning the interaction coupling constant $\mathrm{U} = C\mathrm{N}^{-\alpha}$ we assume $\alpha > 0$ and $C > 0$.

The discrete (one-particle) Gross-Pitaevskii energy functional on the cube Λ_L is defined by

$$\mathrm{E}_{\omega,L}[\varphi] = \langle \mathrm{H}^{\mathrm{N}}_{\omega,\mathrm{L}}\varphi, \varphi\rangle + \mathrm{U}N\|\varphi\|_4^4.$$

The ground state φ^{GP} minimizes the discrete Gross-Pitaevskii energy functional, i.e.

$$\mathrm{E}^{\mathrm{GP}}_{\omega,L} = \mathrm{E}_{\omega,L}[\varphi^{\mathrm{GP}}] = \min_{\substack{\varphi\in l^2(\Lambda_L)\\ \|\varphi\|_2=1}} \mathrm{E}_{\omega,L}[\varphi].$$

$\mathrm{E}^{\mathrm{GP}}_{\omega,L}$ denotes the ground state energy of the discrete GP-functional.

The result

- In the context defined by the discrete Gross-Pitaevskii model we want to prove the following result:

Theorem (Condensation in the single particle ground state)

Denote by φ_0 the single particle ground state of $\mathrm{H}^N_{\omega,L}$ and by φ^{GP} the Gross-Pitaevskii ground state. Assume we can apply the Aizenman-Molchanov technique and the interaction coupling constant satisfies

$$U = o(\log(L)^{-d} L^{-2d}).$$

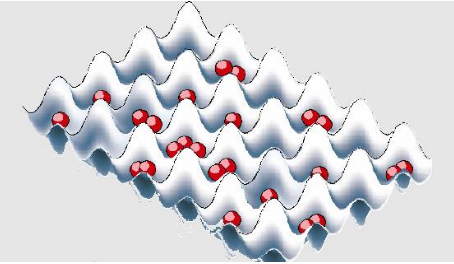
Then we obtain for all $\epsilon > 0$

$$\lim_{L\to\infty} \mathbb{P}[|\langle \varphi_0, \varphi^{\mathrm{GP}}\rangle|^2 \geq 1-\epsilon] = 1,$$

i.e. in the thermodynamic limit and under the assumptions above the Gross-Pitaevskii ground state will be almost surely arbitrary close to the single particle ground state.

Physical motivation

- To appreciate the result maybe some comments about the physical background of the discrete Gross-Pitaevskii model and its relationsship to Bose-Einstein condensation are helpful.
- The discrete Gross-Pitaevskii energy functional is motivated by recent experiments with weakly interacting Bose gases in an optical lattice (see for example [Bloch et al.]).



The ground state of a weakly interacting Bose gas

- Fundamental objects of interest are the ground state density and energy, i.e.

$$\mathcal{E}^{\mathrm{QM}}(\Phi^{\mathrm{QM}}) = \min_{\Phi \in L^2(\Lambda^N)} \langle \Phi, [\sum_{i=1}^{N} \{-\Delta_i + V(x_i)\} + \sum_{1 \le i < j \le N} v(|x_i - x_j|)]\Phi \rangle, \tag{1}$$

  where the optical lattice is modeled by assuming a background potential V as presented on the last slide.
- Assuming a weak interaction limit of the interaction potential $v(x, y)$ the ground state energy (1) can be linked to the discrete Gross-Pitaevskii model in two ways.
- The first approach apply the continuous N-particle Gross-Pitaevskii approximation

$$\mathcal{E}^{\mathrm{GP}}[\varphi^{\mathrm{GP}}] = \min_{\|\varphi\|_2^2 = 1} \int_{\mathbb{R}^3} \left(N|\nabla \varphi(x)|^2 + NV|\varphi(x)|^2 + 4N^2 \pi \mu a |\varphi(x)|^4\right) dx,$$

  to obtain a mean field approximation of the many Boson problem, e.g. in three dimensions one has [Lieb et al.]

$$\lim_{N \to \infty} \frac{\mathcal{E}^{\mathrm{QM}}[\Phi^{\mathrm{QM}}]}{\mathcal{E}^{\mathrm{GP}}[\varphi^{\mathrm{GP}}]} = 1.$$

- The second step is a tight binding approximation of the continuous Gross-Pitaevskii energy functional [Smerzi, Trombettoni].

- Another way to end up with the discrete Gross-Pitaevskii model starts with a discretization of

$$\mathcal{E}^{\mathrm{QM}}(\Phi^{\mathrm{QM}}) = \min_{\Phi \in L^2(\Lambda^N)} \langle \Phi, [\sum_{i=1}^{N} \{-\Delta_i + V(x_i)\} + \sum_{1 \le i < j \le N} v(|x_i - x_j|)] \Phi \rangle,$$

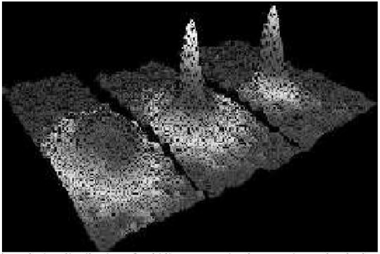
  obtaining the standard description of optical lattices using the Bose - Hubbard - Hamiltonian

$$H = - \sum_{|n-n'|=1} c_n^\dagger c_{n'} + \sum_n (\sigma V_n - \mu) n_n + \frac{1}{2} U \sum n_n^2$$

  where $c_n^\dagger$, c_n are bosonic creation and annihilation operators and n_n gives the particle number at site n. (see for example the survey article [Bloch et al.] and references therein).
- Doing as a second step a mean field approximation one also obtain the discrete Gross-Pitaevskii energy functional [Gunn-Lee] .

Bose-Einstein condensation

- One motivation to study Bose gases is Bose-Einstein condensation, i.e. the phenomenon that a single particle level has a macroscopic occupation (a non-zero density in the thermodynamic limit) [Lieb et al.] .



- Introduced in the context of an ideal Bose gas, it was due to naturally arising interactions a difficult problem to realize Bose-Einstein condensation experimentally [Ketterle, van Druten].

Bose-Einstein condensation

- To formalize the concept of a macroscopic occupation of a single particle state we remember the definition of the one-particle density matrix in the continuous case [Lieb et al.], i.e. the operator on $L^2(\mathbb{R}^3)$ given by the kernel

$$\gamma(x,x') = N \int \Phi^{\mathrm{QM}}(x, x_2, \ldots, x_N) \Phi^{\mathrm{QM}}(x', x_2, \ldots, x_N) \prod_{j=2}^{N} dx_j$$

  with the normalized ground state wave function Φ^{QM} of the multi Boson Hamiltonian.
- BEC in the ground state is then defined that γ has an eigenvalue of order N in the thermodynamic limit.

Bose-Einstein condensation

- Remembering that for the ideal Bose gas the multi-particle ground state can be represented as a product

$$\Phi^{\mathrm{QM}}(x_1, \dots, x_N) = \prod_{i=1}^{N} \varphi_0(x_i)$$

  of the single particle ground state φ_0, the one-particle density matrix becomes

$$\gamma(x, x') = \ N\, \varphi_0(x)\varphi_0(x'),$$

  thus the definition of BEC above is natural.

- The situation in the Gross-Pitaevskii-Limit is close to the situation for the ideal Bose gas [Lieb et al.]. The one-particle density matrix is asymptotically given by

$$\gamma(x, x') \overset{N \to \infty}{\sim} \ N\, \varphi^{\mathrm{GP}}(x)\varphi^{\mathrm{GP}}(x'). \tag{2}$$

  Physically the content of (2) is that all Bose particles will condensate in the GP ground state motivating the definition of complete (or 100%) BEC in [Lieb et al.].

the fine structure of the Gross-Pitaevskii ground state

- Physically the content of the Gross-Pitaevskii approximation is:
  - All Bose particles will condensate in the Gross-Pitaevskii ground state.
  - $\rightarrow$ In some sense this is a single particle state and this is the motivation of the definition of complete (or 100%) BEC in [Lieb et al.]

- The purpose of the present talk is to analyze the fine structure of the Gross-Pitaevskii ground state compared with the eigenstates of the single particle Hamiltonian.

- Under the assumption of a random background potential we want to understand how φ^{GP} is related to the eigenstates of the single particle Hamiltonian.

- More familar is this problem in the following two settings.

Bosons trapped by a potential tending to ∞

- If the Bosons are trapped by a potential with $\liminf_{|x|\to\infty} V(x) = \infty$, the spectral properties of the single particle are invariant in the thermodynamic limit.

$\Rightarrow$ The discrete spectrum and the strictly positive distance between the first two eigenvalues do not depend on the thermodynamic limit.

$\Rightarrow$ Assuming $Na \to 0$ in the continuous setting, respectively $NU \to 0$ in the context of the discrete Gross-Pitaevskii model the interaction energy is a small perturbation of the single particle energy functional.

$\Rightarrow$ In this situation it is natural, that in the thermodynamic limit φ^{GP} and the single particle ground state φ_0 coincide

Bosons confined to a cube without a background potential.

- A complementary situation is given if the Bosons are confined to a cube Λ_L with $|\Lambda_L| \to \infty$ but without a background potential.

$\Rightarrow$ Assuming $\rho = N/L^3$ and $g = Na/L$ in the limit $N \to \infty$ one can prove [Lieb et al.]

$$\lim_{N\to\infty} \frac{1}{N} \frac{1}{L^3} \iint \gamma(x, y) dx dy = 1,$$

$\Rightarrow$ BEC in the normalized single particle ground state $\varphi_0 = L^{-d/2} \chi_{\Lambda_L}$.

$\Rightarrow$ In this context

$$g = \frac{Na}{L} = \frac{\rho a}{1/L^2}$$

is the natural interaction parameter since " in the GP limit the interaction energy per particle is of the same order of magnitude as the energy gap in the box, so that the interaction is still clearly visible". [Lieb et al.]

Bosons trapped in a random background potential

- We want to understand the natural interaction parameter in a random media, s.t. the Gross-Pitaevskii ground state is close to the ground state of the single particle Hamiltonian as it is suggested by the situation in the ideal Bose gas.
- As we will see the setting of Bosons trapped in a random potential is not really comparable to the two situations described above.
  - Under the assumptions defined at the beginning of this publication one has close to $\inf \sigma(H_\omega) = 0$ the Lifshitz regime, i.e. one has pure point spectrum and localized eigenfunctions.

  $\Rightarrow$ In contrast to the situation with vanishing potential the eigenstates close to the bottom of the spectrum are localized in a small part of Λ_L, i.e. the interaction energy will be larger then in the case of the homogenuous Bose gas.
  - Furthermore in the thermodynamic limit the difference between the first two eigenvalues will tend to zero. This difference will almost surely be smaller than for the homogenuous Bose gas.
- Combining these two observations explains the interaction parameter $U = o(\log(L)^{-d} L^{-2d})$.

Strategy to prove the result

- The key idea to prove our result is:

  The distance between the first two single particle eigenvalues is in the thermodynamic limit almost surely larger than a possible energy gain with respect to the interaction term when the GP - ground state is a superposition of more than one single particle eigenstate.

$\Rightarrow$ To prove our result two ingredients are necessary:

- We need upper bounds of the interaction term, i.e. we have to estimate the $\|.\|_4$- norm of the single particle ground state φ_0.
  - We have to control almost surely the single particle ground state energy.
  - We apply a version of the uncertainity principle to estimate $\|\varphi_0\|_4$.
- We have to estimate the probability that the distance of the first two single particle eigenvalues is small.
  - We first exclude the situation where the localization center of first two eigenstates are close together.
  - If the localization center are relatively far away it is possible to decouple the eigenstates and to argue like in the i.i.d. case.

Lemma

Denote by φ_0 the single particle ground state. Then there is a constant $C > 0$ such that

$$\mathbb{P}\left[\|\nabla\varphi_0\|^2 \leq C(\log L)^{-2/d}\|\varphi_0\|^2\right] \geq 1 - o(L^{-1}).$$

Idea of proof:

- Assume $\{B_j\}$ is a decomposition of Λ_L in cubes of side length ℓ.
- Introducing Dirichlet boundary conditions we obtain with $\mathrm{E}_0^{\mathrm{N}}[\omega, \mathrm{L}] = \inf\sigma(\mathrm{H}^N_{\omega,L})$ and $\mathrm{E}_0^{\mathrm{D}}[\omega, \ell, \mathrm{j}] = \inf\sigma(\mathrm{H}^D_{\omega,\ell,B_j})$ the estimate

$$\mathrm{E}_0^{\mathrm{N}}[\omega, \mathrm{L}] \leq \inf_j \mathrm{E}_0^{\mathrm{D}}[\omega, \ell, \mathrm{j}],$$

respectively

$$\begin{aligned}\mathbb{P}[\mathrm{E}_0^{\mathrm{N}}[\omega, \mathrm{L}] \leq E] &\geq \mathbb{P}[\inf_j \mathrm{E}_0^{\mathrm{D}}[\omega, \ell, \mathrm{j}] \leq E] \\ &= 1 - \mathbb{P}[\mathrm{E}_0^{\mathrm{D}}[\omega, \ell, \mathrm{j}] > E]^{(L/\ell)^d} \\ &= 1 - \left(1 - \mathbb{P}[\mathrm{E}_0^{\mathrm{D}}[\omega, \ell, \mathrm{j}] \leq E]\right)^{(L/\ell)^d}.\end{aligned}$$

Lower estimate of $\mathbb{P}\left[\| \bigtriangledown \varphi_0\|^2 \leq C(\log L)^{-2/d}\|\varphi_0\|^2\right]$

- Choosing $E = \ell^{-2}$ and applying the Lifshitz estimate

$$\mathbb{P}[\mathrm{E}_0^{\mathrm{D}}[\omega, \ell, \mathrm{j}] \leq E] \geq C_1 \exp(-C_2 \ell^d)$$

  we obtain

$$\begin{aligned}
\mathbb{P}[\mathrm{E}_0^{\mathrm{N}}[\omega, \mathrm{L}] \leq E] &\geq 1 - \exp\left((L/\ell)^d \log(1 - \mathbb{P}[\mathrm{E}_0^{\mathrm{D}}[\omega, \ell, \mathrm{j}] \leq E])\right) \\
&\geq 1 - \exp\left(-C(L/\ell)^d \mathbb{P}[\mathrm{E}_0^{\mathrm{D}}[\omega, \ell, \mathrm{j}] \leq E]\right) \\
&\geq 1 - \exp\left(-C_1 (L/\ell)^d \exp(-C_2 \ell^d)\right).
\end{aligned}$$

- Choosing $\ell = (0.5 C_2^{-1} \log L)^{1/d}$ we have

$$C_1 (L/\ell)^d \exp(-C_2 \ell^d) = O\left(\frac{L^{d-0.5}}{\log L}\right),$$

  respectively

$$\mathbb{P}[\mathrm{E}_0^{\mathrm{N}}[\omega, \mathrm{L}] \leq E] \geq 1 - \exp\left(-C_1 (L/\ell)^d \exp(-C_2 \ell^d)\right) \geq 1 - o(L^{-1}).$$

- Finally we obtain

$$\begin{aligned}
1 &\geq \mathbb{P}\left[\| \bigtriangledown \varphi_0\|^2 \leq C(\log L)^{-2/d}\|\varphi_0\|^2\right] \\
&\geq \mathbb{P}\left[\mathrm{E}_0[\omega, \mathrm{L}]\|\varphi_0\|^2 \leq C(\log L)^{-2/d}\|\varphi_0\|^2\right] \\
&\geq 1 - o(L^{-1}).
\end{aligned}$$

Estimate of the Gross-Pitaevskii ground state energy

Lemma (Approximation lemma [Klopp])

Assume $u \in l^2(\Lambda)$ *with* $\|u\|_2^2 = 1$ *and* $\|\nabla u\|_2^2 \leq \epsilon$, $\epsilon > L^{-1}$. *Furthermore denote by* $\{B_j\}$ *a decomposition of* Λ *in cubes of side length* $\epsilon^{-1/2+\theta}$, $\theta < 0.5$. *Then there is* $v \in l^2(\Lambda)$ *and* $\delta > 0$, *s.t.*

- $\|v - u\|_2 \leq \epsilon^{\delta}$,
- $\|v\|_2 = 1$,
- $v = \sum_j c_j \chi_{B_j}$.

Lemma

The Gross-Pitaevskii ground state energy can be estimated by

$$\mathbb{P}\left[\mathrm{E}^{\mathrm{GP}}_{\omega,L} \leq \mathrm{E}_0[\omega, \mathrm{L}] + CUN(\log L)^{-2/d\delta}\right] \geq 1 - o(L^{-1}).$$

Idea of proof:

- We have

$$1 = \|v\|_2^2 = \sum_j |c_j|^2 |B_j| = \epsilon^{-d(1/2-\theta)} \sum_j |c_j|^2$$

$\implies$ $|c_j| \leq \epsilon^{d/2(1/2-\theta)}$

$\implies$ $\|v\|_4^4 = \sum_j |c_j|^4 |B_j| \leq \epsilon^{d(1/2-\theta)} \sum_j |c_j|^2 |B_j| = \epsilon^{d(1/2-\theta)}$

- Combining the estimate above and Klopp's approximation lemma we obtain

$$\begin{aligned} \|u\|_4 &\leq \|u-v\|_4 + \|v\|_4 \\ &\leq \sqrt{\|u-v\|_2} + \epsilon^{d(1/2-\theta)} \\ &\leq \epsilon^{\delta/2} \|u\|_2 + \epsilon^{d(1/2-\theta)} \\ &\leq C\epsilon^{\widetilde{\delta}}. \end{aligned}$$

- Finally we obtain

$$\begin{aligned} 1 &\geq \mathbb{P}\left[\mathrm{E}^{\mathrm{GP}}_{\omega,L} \leq \mathrm{E}_0[\omega,\mathrm{L}] + \frac{C(\log L)^{-2/d\delta}}{L^d} \right] \\ &\geq \mathbb{P}\left[\| \nabla \varphi_0\|^2 \leq C(\log L)^{-2/d} \|\varphi_0\|^2 \right] \geq 1 - o(L^{-1}). \end{aligned}$$

Lemma (Localization lemma [Klopp])

Assume H_ω as defined in the introduction, $\delta > 0$ and let $I \subset \sigma(\mathrm{H}_\omega)$ be a compact interval, s.t. the assumptions of the Aizenman-Molchanov technique are satisfied, i.e. I is a subset of the localized regime.
Choose a cube Λ and suppose that $\varphi_{n,\omega}$ is an eigenvector of $\mathrm{H}_\omega|_\Lambda$ associated to $E_{n,\omega} \in I$, denote by $x_n(\omega) \in \Lambda$ its localization center, by α its localization length and define

$$\Omega_{I,\delta} = \{ \omega \in \Omega : \forall \varphi_{n,\omega} \text{ associated to } E_{n,\omega} \in I, \exists x_n(\omega) \in \Lambda \text{ s.t. } \forall x \in \Lambda,$$

$$|\varphi_{n,\omega}(x)| \leq \frac{C}{\delta^2} |\Lambda|^{7/2} \exp(-\alpha|x - x_n(\omega)|) \}.$$

Then we obtain for all $\delta > 0$ the estimate

$$\mathbb{P}[\Omega_{I,\delta}] \leq 1 - \delta.$$

Small distance between the localization center of the first two eigenvalues

Lemma

Choose $\eta > 0$ *and* $I = (0, C(\log(L))^{-2/d}]$ *with* L *large enough, s.t. the interval* I *will satisfy the assumptions of the localization lemma. Furthermore choose* $\rho > d$ *and* $\lambda > 0$*, s.t.*

$$L^{5d} \exp\left(-\alpha\lambda \log(L)\right) \;=\; O(L^{-\rho}).$$

The first two eigenvalues of $\mathrm{H}^N_{\omega,L}$ *are denoted by* $\mathrm{E_k}[\omega, \mathrm{L}]$, $k = 1, 2$ *and its localization center by* $x_k(\omega)$, $k = 1, 2$*. Then we can estimate*

$$\mathbb{P}\left[\mathrm{E}_1[\omega,\mathrm{L}] - \mathrm{E}_0[\omega,\mathrm{L}] \le \eta L^{-d}, |x_0(\omega) - x_1(\omega)| \le \lambda \log(L)\right]$$
$$= O(C\eta^2(\log(L))^{-2/d}\log(L)^2).$$

Small distance between the localization center of the first two eigenvalues

Idea of proof:

Step 1: Decomposition of the random potentials w.r.t. the loc. center:

- Using $|x_0(\omega) - x_1(\omega)| \le \lambda \log(L)$ we can define a family of decompositions $\{D_i^j\}$, $j = 1, .., M(d)$, $i = 1, .., L^d (5\lambda \log(L))^{-d}$ of cubes of side length $5\lambda \log(L)$ such that for each $j = 1, .., M(d)$

$$\Lambda_L = \bigcup_{i=1}^{L^d (5\lambda \log(L))^{-d}} D_i^j$$

and with

$$\widetilde{\Omega} = \{\omega \in \Omega : |x_0(\omega) - x_1(\omega)| \le \lambda \log(L), k = 0, 1\}$$

$$\Omega_i^j = \{\omega \in \widetilde{\Omega} : x_k(\omega) \in D_i^j, dist(x_k, \partial D_i^j) \ge \lambda \log(L), k = 0, 1\}$$

the decomposition

$$\widetilde{\Omega} = \bigcup_{j=1}^{M(d)} \bigcup_{i=1}^{(5\lambda \log(L))^{-d} L^d} \Omega_i^j$$

is satisfied.

- Then we can estimate

$$\mathbb{P}\left[\mathrm{E}_1[\omega, \mathrm{L}] - \mathrm{E}_0[\omega, \mathrm{L}] \le \eta L^{-d}, |x_0(\omega) - x_1(\omega)| \le \lambda \log(L)\right]$$

$$\le \sum_{j=1}^{M(d)} \sum_{j=1}^{(5\lambda \log(L))^{-d} L^d} \mathbb{P}\left[\omega \in \Omega_i^j : \mathrm{E}_1[\omega, \mathrm{L}] - \mathrm{E}_0[\omega, \mathrm{L}] \le \eta L^{-d}\right].$$

Small distance between the localization center of the first two eigenvalues

Step 2: Approximation of the eigenvalues and eigenfunctions on the local cube:

- Assume $\omega \in \Omega_i^j$ and $\varphi_{k,\omega}$ are the eigenfunctions corresponding to $\mathrm{E_k}[\omega, \mathrm{L}]$, $k \in \{0,1\}$. Restricting $\varphi_{k,\omega}$ to D_i^j by

$$\widetilde{\varphi}_{k,\omega} = \chi_{D_i^j} \varphi_{k,\omega}, \qquad k \in \{0,1\}.$$

  and as a consequence of localization lemma and the definition of Ω_i^j we can approximate

$$\|\widetilde{\varphi}_{k,\omega}\|_{l^2(D_i^j)} \geq 1 - O(L^{-\rho}),$$

$$\langle \widetilde{\varphi}_{0,\omega}, \widetilde{\varphi}_{1,\omega} \rangle = O(L^{-\rho})$$

  and

$$\mathrm{H}_\omega|_{D_i^j} \widetilde{\varphi}_{k,\omega} = \mathrm{E_k}[\omega, \mathrm{L}] \widetilde{\varphi}_{k,\omega} + O(L^{-\rho}).$$

- Applying these approximations to the estimate of the last slide and denoting by Λ_ℓ a cube of side length $\ell = 5\lambda \log(L)$ we are close to the i.i.d. situation and we are able to estimate

$$\mathbb{P}\left[\mathrm{E}_1[\omega, \mathrm{L}] - \mathrm{E}_0[\omega, \mathrm{L}] \leq \eta L^{-d}, |x_0(\omega) - x_1(\omega)| \leq \lambda \log(L)\right]$$

$$= \frac{C L^d}{log(L)} \mathbb{P}\left[\mathrm{E}_1[\omega, \ell] - \mathrm{E}_0[\omega, \ell]) \leq \eta L^{-d}, x_k(\omega) \in \Lambda_\ell, k = 0, 1\right].$$

Small distance between the localization center of the first two eigenvalues

Step 3: Application of the Minami estimate:

- Decompose

$$I = (0, \lambda \log(L)^{-2/d}] = \bigcup_{k=1}^{K} H_k^m \qquad m = 1, 2$$

s.t. $|H_k^m| = 2L^{-d}$, $H_k^m \cap H_{\tilde{k}}^m = \emptyset$ if $k \neq \tilde{k}$, $K \leq (\lambda \log(L))^{-2/d} L^d$ and $\exists k \in \{1, .., K\}$, s.t.$\mathrm{E_i}[\omega, \tilde{\ell}] \in H_k^m, i = 0, 1, k = 1, 2$

- By applying the Minami estimate we obtain

$$\begin{aligned}
&\mathbb{P}[\mathrm{E}_1[\omega, \ell] - \mathrm{E}_0[\omega, \ell] \leq \eta L^{-d}] \\
&\leq \sum_{m=1}^{2} \sum_{k=1}^{K} \mathbb{P}[\mathrm{E_i}[\omega, \ell] \in H_k^m, i = 0, 1] \\
&\leq \sum_{m=1}^{2} \sum_{k=1}^{K} C(|H_k^m| \ell^d)^2 \\
&= O(C(\log(l))^{-2/d} L^d (\eta L^{-d} \ell^d)^2) \\
&= O(C\eta^2 (\log(L))^{-2/d} \log(L)^2 L^{-d}),
\end{aligned}$$

respectively

$$\begin{aligned}
&\mathbb{P}\left[\mathrm{E}_1[\omega, \mathrm{L}] - \mathrm{E}_0[\omega, \mathrm{L}] \leq \eta L^{-d}, |x_0(\omega) - x_1(\omega)| \leq \lambda \log(L)\right] \\
&= O(C\eta^2 (\log(L))^{-2/d} \log(L)^2).
\end{aligned}$$

Large distance between the localization center of the first two eigenvalues

Lemma

Under the assumptions of the last Lemma we can estimate

$$\mathbb{P}\left[\mathrm{E}_1[\omega,\mathrm{L}] - \mathrm{E}_0[\omega,\mathrm{L}] \leq \eta L^{-d}, |x_0(\omega) - x_1(\omega)| > \lambda \log(L)\right] = O(\eta).$$

- Define a decomposition like in the last Lemma, s.t. we can apply the approximation argument above.

$\Rightarrow$ We have to estimate $\mathbb{P}[\mathrm{E}_1[\omega,\ell] - \mathrm{E}_0[\omega,\ell] \leq \epsilon L^{-d}]$ where $\mathrm{E}_1[\omega,\ell]$ is an eigenvalue with an eigenfunction localized in Λ_1, $\mathrm{E}_2[\omega,\ell]$ is an eigenvalue with an eigenfunction located in Λ_2 and $\Lambda_1 \cap \Lambda_2 = \emptyset$.

$\Rightarrow$ We can apply the Wegner estimate and obtain

$$\begin{aligned}
&\mathbb{P}[\mathrm{E}_1[\omega,\ell] - \mathrm{E}_0[\omega,\ell] \leq \eta L^{-d}] \\
&\quad \leq \mathbb{E}\left[\mathbb{P}[\mathrm{E}_1[\omega,\ell] - \mu \leq \eta L^{-d}] | \mathrm{E}_0[\omega,\ell] = \mu\right] \\
&\quad = O(\eta L^{-d}).
\end{aligned}$$

- Combining this estimate with the factor L^d resulting from the approximation argument we obtain the desired bound.

Distance of the first two eigenvalues

Corollary

Under the assumptions of the last lemmas and with $\eta = \log(L)^{-d}$ *we obtain*

$$\mathbb{P}\left[\mathrm{E}_1[\omega, \mathrm{L}] - \mathrm{E}_0[\omega, \mathrm{L}] \leq \eta L^{d}\right] = O(\log(L)^{-d})$$

- Combining the two estimates above we finally arrive at

$$\begin{aligned}
&\mathbb{P}\left[\mathrm{E}_1[\omega, \mathrm{L}] - \mathrm{E}_0[\omega, \mathrm{L}] \leq \eta L^{-d}\right] \\
&= \mathbb{P}\left[\mathrm{E}_1[\omega, \mathrm{L}] - \mathrm{E}_0[\omega, \mathrm{L}] \leq \eta L^{-d} \wedge |x_0(\omega) - x_1(\omega)| \geq \lambda \log(L)\right] \\
&\quad + \mathbb{P}\left[\mathrm{E}_1[\omega, \mathrm{L}] - \mathrm{E}_0[\omega, \mathrm{L}] \leq \eta L^{-d} \wedge |x_0(\omega) - x_1(\omega)| < \lambda \log(L)\right] \\
&\leq O(\eta) + O(C\eta^2 (\log(L))^{-2/d} \log(L)) \\
&= O(\log(L)^{-d}).
\end{aligned}$$

Proof of the Theorem

- Defining $\pi_0 = |\varphi_0\rangle\langle\varphi_0|$ and applying the definition of the ground state we can estimate

$$\mathrm{E}_1[\omega, \mathrm{L}]\|(1-\pi_0)\varphi^{\mathrm{GP}}\| + \mathrm{E}_0[\omega, \mathrm{L}]\|\pi_0\varphi^{\mathrm{GP}}\| \\ \leq \mathrm{E}^{\mathrm{GP}}_{\omega,L}\|(1-\pi_0)\varphi^{\mathrm{GP}}\| + \mathrm{E}^{\mathrm{GP}}_{\omega,L}\|\pi_0\varphi^{\mathrm{GP}}\|,$$

  respectively

$$\left(\mathrm{E}_1[\omega, \mathrm{L}] - \mathrm{E}^{\mathrm{GP}}_{\omega,L}\right)\|(1-\pi_0)\varphi^{\mathrm{GP}}\| \leq \left(\mathrm{E}^{\mathrm{GP}}_{\omega,L} - \mathrm{E}_0[\omega, \mathrm{L}]\right)\|\pi_0\varphi^{\mathrm{GP}}\|.$$

- As a consequence of the last results we can assume with probability tending to 1 that

$$\mathrm{E}_1[\omega, \mathrm{L}] - \mathrm{E}^{\mathrm{GP}}_{\omega,L} \geq \mathrm{E}_1[\omega, \mathrm{L}] - \mathrm{E}_0[\omega, \mathrm{L}] \geq L^{-d}\log(L)^{-d}$$

  and

$$\mathrm{E}^{\mathrm{GP}}_{\omega,L} - \mathrm{E}_0[\omega, \mathrm{L}] \leq CUN(\log L)^{-2/d\delta}.$$

- We obtain

$$\|(1-\pi_0)\varphi^{\mathrm{GP}}\| \leq CUNL^d(\log L)^{-2/d\delta}\log(L)^d\|\pi_0\varphi^{\mathrm{GP}}\|,$$

Proof of the Theorem

respectively

$$
\begin{aligned}
|\langle \varphi_0, \varphi^{\mathrm{GP}} \rangle|^2 &= \|\pi_0 \varphi^{\mathrm{GP}}\| \\
&= 1 - \|\pi_0 \varphi^{\mathrm{GP}}\| \\
&\geq 1 - CUNL^d \log(L)^d (\log L)^{-2/d\delta} \\
&\to 1
\end{aligned}
$$

by applying the assumption $U = o(\log(L)^{-d} L^{-2d})$.