Asymptotics of Return Probabilities of Random Walks on Free Products of Lattices

ELISABETTA CANDELLERO & LORENZ A. GILCH
Graz University of Technology, Austria
email: candellero@tugraz.at, gilch@tugraz.at

TUG

Department of Mathematical Structure Theory

The n-step return probabilities of nearest neighbour random walks on the free product $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2} * \ldots * \mathbb{Z}^{d_m}$ have one of the following asymptotic behaviours:

$$\mu^{(\mathbf{2n})}(\mathbf{e}) \sim \begin{cases} C_1 \cdot \varrho^{2n} \cdot n^{-\delta/2}, & \text{for one } 5 \leq \delta \in \{d_1, \ldots, d_m\} \\ C_2 \cdot \varrho^{2n} \cdot n^{-3/2}. \end{cases} \quad (\varrho \text{ is the corresponding spectral radius})$$

I. Random Walks on $\Gamma := \mathbb{Z}^{d_1} * \mathbb{Z}^{d_2} * \ldots * \mathbb{Z}^{d_m}$

Definitions and Notation

Free Product: $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2} * \ldots * \mathbb{Z}^{d_m}$ ($d_i \in \mathbb{N}$) is the group of all words of the form $x_1 x_2 \ldots x_n$, where the $x_j \in \bigcup_{i=1}^m \mathbb{Z}^{d_i} \setminus \{\mathbf{0}\}$ and two consecutive letters are not from the same lattice. The identity is the empty word e.

(Nearest Neighbour) Random Walk on Γ is governed by $\mu := \alpha_1 \mu_1 + \ldots + \alpha_m \mu_m$, where μ_i are irreducible probability measures on $\mathbb{Z}^{d_i}$ with $\mathrm{supp}(\mu_i) = \{$set of natural generators of $\mathbb{Z}^{d_i}\}$ and $\alpha_i > 0$ with $\sum_{i=1}^m \alpha_i = 1$.

Return Probabilities: $\mu^{(n)}(e) =$ probability of returning to e in n steps.

Green Functions: given $z \in \mathbb{C}$, we define

$$G_i(z) := \sum_{n=0}^{\infty} \mu_i^{(n)}(\mathbf{0}) z^n, \quad G(z) := \sum_{n=0}^{\infty} \mu^{(n)}(e) z^n.$$

The corresponding radii of convergence are $\mathbf{r}_i$ and $\mathbf{r}$. We denote by $\varrho = 1/\mathbf{r}$ the spectral radius. There are functions Φ_i, $i \in \{1, \ldots, m\}$, and Φ (see Woess (4)) such that

$$G_i(z) = \Phi_i\big(zG_i(z)\big) \quad \text{and} \quad G(z) = \Phi\big(zG(z)\big).$$

Φ_i and Φ are analytic in an open neighbourhood of $[0, \theta_i)$, $[0, \theta)$ respectively, where $\theta_i := \mathbf{r}_i G_i(\mathbf{r}_i)$ and $\theta := \mathbf{r} G(\mathbf{r})$. Moreover, $\bar{\theta} := \min\{\theta_i/\alpha_i \mid 1 \leq i \leq m\}$. Furthermore, we define

$$\Psi_i(t) := \Phi_i(t) - t\Phi_i'(t) \quad \text{and} \quad \Psi(t) := \Phi(t) - t\Phi'(t) = 1 + \sum_{i=1}^m \big(\Psi_i(\alpha_i t) - 1\big).$$

Expansion of the Green Function on $\mathbb{Z}^d$

Proposition 1. *For $d \in \mathbb{N}$, the Green function on $\mathbb{Z}^d$ has an expansion of the form:*

$$G_d(z) = \begin{cases} f(z) + g(z)(\mathbf{r}_d - z)^{(d-2)/2}, & \text{if } d \text{ is odd}, \\ f(z) + g(z)(\mathbf{r}_d - z)^{(d-2)/2} \log(\mathbf{r}_d - z), & \text{if } d \text{ is even}, \end{cases}$$

where r_d is the radius of convergence and $f(z), g(z)$ are analytic in a neighbourhood of $z = \mathbf{r}_d$ and $g(\mathbf{r}_d) \neq 0$.

Remark: For the simple random walk on $\mathbb{Z}^d$ a proof can be found in Cartwright (1) and Woess (4).

II. If $m = 2$ we distinguish 3 cases, according to the sign of $\Psi(\bar{\theta})$

The Case $\Psi(\bar{\theta}) < 0$ on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$

The asymptotic behaviour of the return probabilities of a random walk on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ is

$$\mu^{(2n)}(e) \sim C \cdot \varrho^{2n} n^{-3/2};$$

see Woess (4, Theorem 17.3). The same asymptotic behaviour holds if $d_1, d_2 \leq 4$.

The Case $\Psi(\bar{\theta}) > 0$ on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$

Theorem 2. *Let be $5 \leq d_1 \leq d_2$. Then:*

$$\mu^{(2n)}(e) \sim \begin{cases} C_1 \cdot \varrho^{2n} \cdot n^{-d_1/2}, & \text{if } \alpha_1 \geq \frac{\theta_1}{\theta_1 + \theta_2}, \\ C_2 \cdot \varrho^{2n} \cdot n^{-d_2/2}, & \text{if } \alpha_1 < \frac{\theta_2}{\theta_1 + \theta_2}. \end{cases}$$

Idea of the Proof

There are functions $\xi_1(z)$ and $\xi_2(z)$ such that

$$\alpha_i z G(z) = \xi_i(z) G_i(\xi_i(z)) \quad \text{for } i = 1, 2.$$

It is sufficient to find an expansion of $\xi_1(z)$ and $\xi_2(z)$ in a neighbourhood of $z = \mathbf{r}$, in order to obtain the leading singular term of $G(z)$. With the help of *Darboux's method* we find the proposed asymptotic behaviour.

The Case $\Psi(\bar{\theta}) = 0$ on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$

Theorem 3. *Assume that $\Psi(\bar{\theta}) = 0$. Then we have the following asymptotic behaviour:*

$$\mu^{(2n)}(e) \sim C \varrho^{2n} n^{-3/2}.$$

Remark: For the proof of the theorem, we distinguish whether $\Psi''(\bar{\theta})$ is finite or not. In particular, we show that the function $\Psi(\cdot)$ may have a zero in $\bar{\theta}$ only if $\Psi''(\bar{\theta}) < \infty$.

III. Generalization to $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2} * \ldots * \mathbb{Z}^{d_m}$

Free Products of $m > 2$ lattices

Considering the free product $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2} * \ldots * \mathbb{Z}^{d_m}$ we get inductively one of the following asymptotic behaviours:

$$\mu^{(2n)}(e) \sim \begin{cases} C_1 \cdot \varrho^{2n} \cdot n^{-\delta/2}, & \text{for one } 5 \leq \delta \in \{d_1, \ldots, d_m\} \\ C_2 \cdot \varrho^{2n} \cdot n^{-3/2}. \end{cases}$$

Example: Simple Random Walks on the factors $\mathbb{Z}^{d_1}, \ldots, \mathbb{Z}^{d_m}$

Suppose that the μ_i govern simple random walks on $\mathbb{Z}^{d_i}$. If all the exponents $d_i \geq 5$ are large enough, we always have $\Psi(\bar{\theta}) > 0$. In this case, choosing α_k for any $k \in \{1, \ldots, m\}$ sufficiently large, we get

$$\mu^{(2n)}(e) \sim C_k \cdot \varrho^{2n} \cdot n^{-d_k/2}.$$

Furthermore, one can always choose μ such that we get the asymptotic behaviour

$$\mu^{(2n)}(e) \sim C_0 \cdot \varrho^{2n} \cdot n^{-3/2}.$$

Compare with Cartwright (2) and Woess (4).

References

[1] Cartwright D.I.: Some examples of random walks on free products of discrete groups. *Annali Matematica Pura ed Applicata. Serie IV.* **151** pp. 1–15 (1988).

[2] Cartwright D.I.: On the asymptotic behaviour of convolution powers of probabilities on discrete groups. *Monatshefte für Mathematik.* **107** (4) pp. 287–290 (1989).

[3] Gerl P.: A local central limit theorem on some groups. In *The first Pannonian Symposium on Mathematical Statistics.* Lect. Notes in Statistics **8** pp. 73–82, Springer (1981).

[4] Woess, W.: *Random Walks on Infinite Graphs and Groups*, Cambridge Tracts in Mathematics **138**, Camb. Univ. Press, (2000).

Research supported by Büro für Forschung und Technologie (TU Graz), NAWI-Graz and German Research Foundation (DFG) grant GI 746/1-1.