Random trees and absolutely continuous spectrum

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1. The Laplace operator

Let $G = (V, E)$ be a graph.

The Laplace operator $\Delta = \Delta_G$ on $\ell^2(V)$ be given by

$$(\Delta\varphi)(x) = \sum_{\{x,y\}\in E} (\varphi(x) - \varphi(y))\,.$$

Trees of finitely many cone types

Let $\mathcal{A} = \{1, \ldots, N\}$ be a set of labels and

$$M : \mathcal{A} \times \mathcal{A} \to \mathbb{N}, \quad (j,k) \mapsto m_{j,k}.$$

Construct a tree $T_M = (V, E)$ with vertices labeled by $\mathcal{A}$ in the following way:

(i) Let the root have label $i \in \mathcal{A}$.

(ii) A vertex with label j has exactly $m_{j,k}$ forward neighbors of label k.

The spectrum of Δ on T_M

This construction gives a tree $T_M = (V, E)$ with finitely many cone types, where every cone type has every cone type as forward neighbor.

Theorem. (K., Lenz, Warzel) The spectrum consists of finitely many intervalls

$$\sigma(\Delta_{T_M}) = \sigma_{\rm a.c}(\Delta_{T_M}) \quad \text{and} \quad \sigma_{\rm sing}(\Delta_{T_M}) = \emptyset.$$

Percolating $E \setminus E'$

For each $x \in T_M$ fix $x' \succ x$, such that if x and y have the same label then x' and y' have the same label. Set

$$E' = \{\{x, x'\} \mid x \in V\}.$$

Let $p \in [0,1]$ and θ random variables

$$\theta : E \setminus E' \to \{0,1\}$$

such that $\{\theta(e)\}$ are independent and

$$\mathbb{P}(\theta(e) = 1) = p \quad \text{and} \quad \mathbb{P}(\theta(e) = 0) = 1 - p.$$

The trees T_θ and the operators Δ_θ

Let $T_\theta = (V, E_\theta) \subset T_M$ such that

$$E_\theta = E' \cup \{e \in E \setminus E' \mid \theta(e) = 1\}.$$

Define Δ_θ on $\ell^2(V)$ by

$$(\Delta_\theta \varphi)(x) = \sum_{\{x,y\} \in E_\theta} \varphi(x) - \varphi(y).$$

Absolutely continuous spectrum on T_θ

Theorem. (K., Lenz, Warzel)
Let $I \subset \sigma(\Delta_{T_M}) \setminus \Sigma_0$ closed, where Σ_0 is finite. Then there exists $p_0 < 1$ such that for all $p \in [p_0, 1]$ and almost every θ

$$I \subset \sigma_{\mathrm{a.c}}(\Delta_\theta) \quad \text{and} \quad I \cap \sigma_{\mathrm{sing}}(\Delta_\theta) = \emptyset.$$

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For the presentation of some key ideas of the proof we restrict ourselves to the special case $\mathcal{A} = \{1\}$ and $M = k$, i.e. the k-regular tree.

Some ideas of the proof

Let $T_x \subset T_M$ be the forward cone starting at $x \in T_M$. Denote the Green's function by

$$\begin{aligned} g_x^\theta(z) &= \langle \delta_x, (\Delta_\theta|_{T_x} - z)^{-1}\delta_x\rangle. \\ G_x(z) &= \langle \delta_x, (\Delta|_{T_x} - z)^{-1}\delta_x\rangle. \end{aligned}$$

Define

$$\gamma_x = \frac{|g_x^\theta - G_x|^2}{\Im g_x^\theta \Im G_x}.$$

We want to prove:

$$\mathbb{E}(\gamma_x) \leq C, \quad \text{all } \Re z \in I.$$

Some ideas of the proof II

By the recursion formulas for the Green's function :

(1) If $\theta(e) = 1$ for all $y \succ x$ and $v \succ x'$

$$\gamma_x = \sum_{\{x,y\} \in E} \frac{1}{k} c_y \gamma_y = \sum_{\substack{\{x,y\} \in E, \\ y \neq x'}} \frac{1}{k} c_y \gamma_y + \sum_{\{v,x'\} \in E} \frac{1}{k^2} c_v \gamma_v$$

(2) Otherwise there are $C, c > 0$

$$\gamma_x \leq C \left(\sum_{\substack{\{x,y\} \in E, \\ y \neq x'}} \frac{1}{k} \gamma_y + \sum_{\{v,x'\} \in E} \frac{1}{k^2} \gamma_v \right) + c.$$

Some ideas of the proof III

$$
\begin{aligned}
\mathbb{E}(\gamma_x \mid (1)) &= \mathbb{E}\left(\sum_{y \succ x} \frac{c_y \gamma_y}{k} + \sum_{v \succ x'} \frac{c_v \gamma_v}{k^2} \mid (1)\right) \\
&\overset{(!)}{\leq} (1-\delta)\mathbb{E}\left(\sum_{y \succ x} \frac{\gamma_y}{k} + \sum_{v \succ x'} \frac{\gamma_v}{k^2} \mid (1)\right) \\
&= (1-\delta)\mathbb{E}\left(\gamma_x\right)
\end{aligned}
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&= (1-\delta)\mathbb{E}\left(\gamma_x\right) \\
\mathbb{E}(\gamma_x \mid (2)) &\leq C\mathbb{E}\left(\sum_{y \succ x} \frac{\gamma_y}{k} + \sum_{v \succ x'} \frac{\gamma_v}{k^2} \mid (2)\right) + c \\
&= C\mathbb{E}\left(\gamma_x\right) + c
\end{aligned}
$$

Some ideas of the proof IV

Let $n = 2k - 1$

$$\begin{aligned}\mathbb{E}(\gamma_x) &= p^n \mathbb{E}(\gamma_x \mid (1)) + (1 - p^n)\mathbb{E}(\gamma_x \mid (2)) \\ &\leq ((1-\delta)p^n + C(1-p^n))\mathbb{E}(\gamma_x) + c.\end{aligned}$$

Hence for $p^n > \frac{1}{1+\delta/(C-1)}$

$$\mathbb{E}(\gamma_x) \leq (1-\delta')\mathbb{E}(\gamma_x) + c.$$

Thus

$$\mathbb{E}(\gamma_x) \leq \frac{c}{\delta'}.$$

What is c_y

Let $y \succ v$

$$c_y = \sum_{x \in S_v} \frac{\Im g_x^\theta}{\sum_{u \in S_x} \Im g_u^\theta} Q_{x,y} \cos \alpha_{x,y},$$

while

$$Q_{x,y} = \frac{\left(\Im g_x^\theta \Im g_y^\theta \Im G_x \Im G_y \gamma_x \gamma_y\right)^{\frac{1}{2}}}{\frac{1}{2}\left(\Im g_x^\theta \Im G_y \gamma_y + \Im g_y^\theta \Im G_x \gamma_x\right)}$$

and

$$\alpha_{x,y} = \arg\left((g_x^\theta - G_x)\overline{(g_y^\theta - G_y)}\right).$$

What is different if $\mathcal{A} \neq \{1\}$

- (1):

$$\gamma_x = \sum_y \frac{\Im G_y}{\sum\limits_{u \in S_x} \Im G_u} c_y \gamma_y + \frac{\Im G_{x'}}{\sum\limits_{u \in S_x} \Im G_u} \sum_v \frac{\Im G_y}{\sum\limits_{u \in S_{x'}} \Im G_u} c_v \gamma_v$$

- (2) analogue
- (!) Instead of all permutations take only those who leave the label invariant. Analysis of the contraction coefficient!
- Peron/Frobenius Theorem