Multifractal analysis of Galton-Watson trees

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University of Bath

joint work with Adam Kinnison (Bath)

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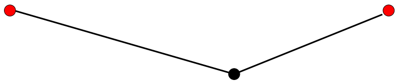

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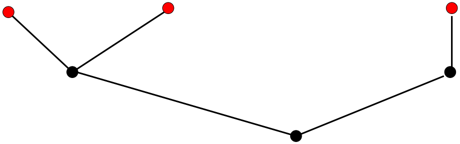

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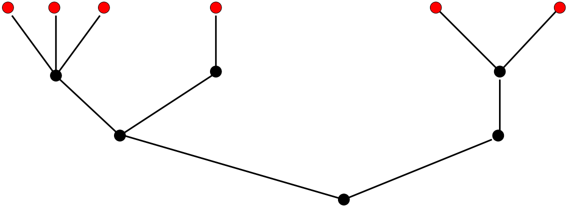

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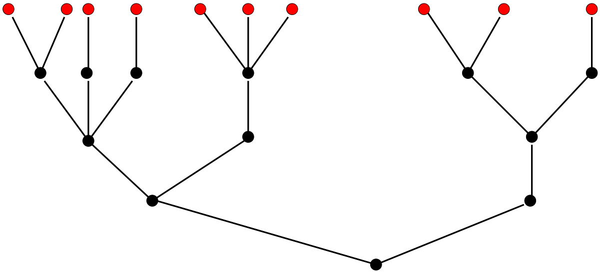

Galton-Watson trees: the boundary

The boundary ∂T of the tree is the collection of all rays, i.e. the infinite paths

$$x = (v_0, v_1, v_2, \ldots)$$

such that $v_0 = \rho$ and v_i is a child of v_{i-1}, for all $i \geq 2$.

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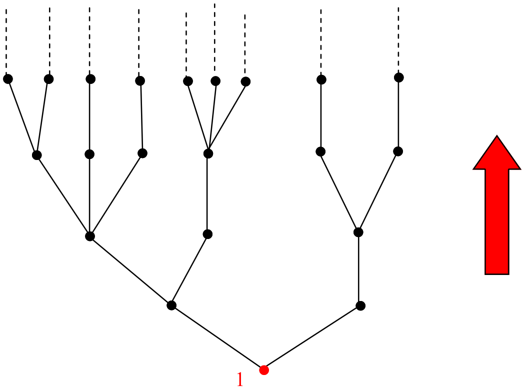
Probability measures on ∂T correspond to (transient) random walks on the tree.

Two interesting measures on the boundary

Visibility or harmonic measure ν



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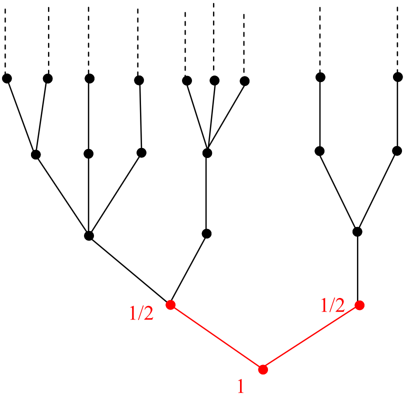

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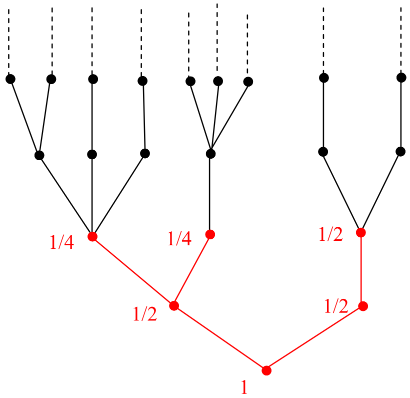

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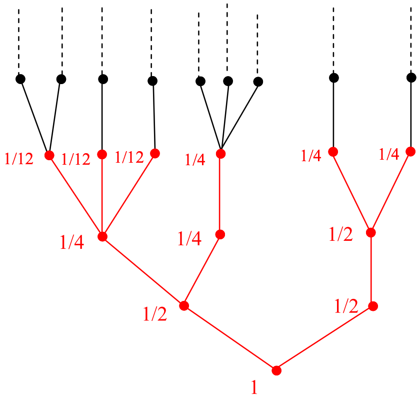

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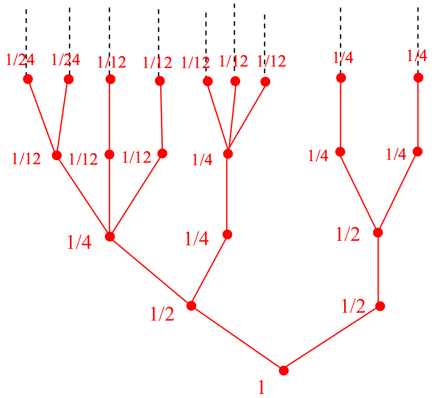

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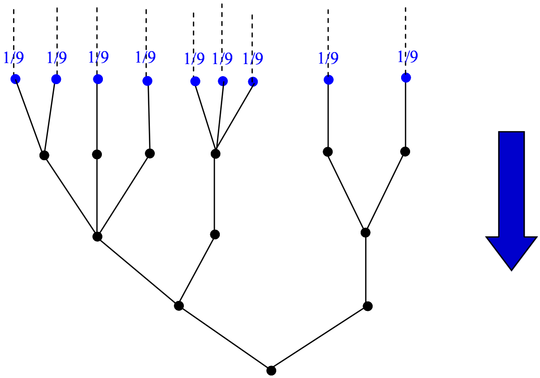

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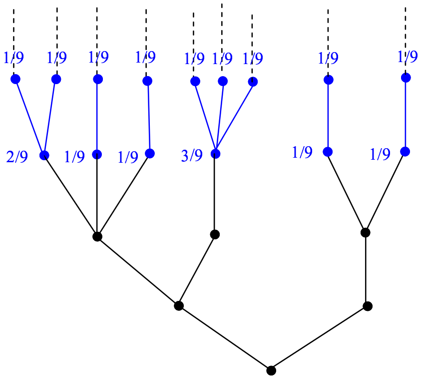

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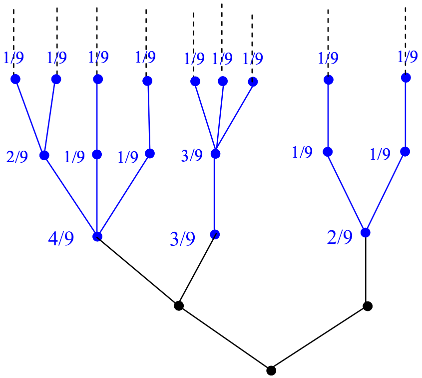

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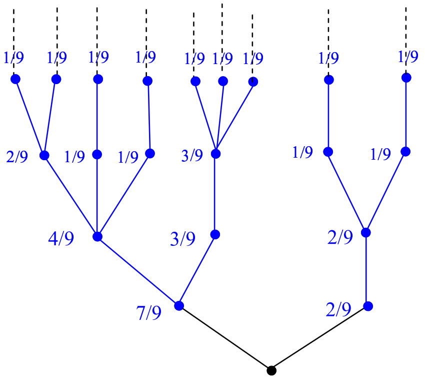

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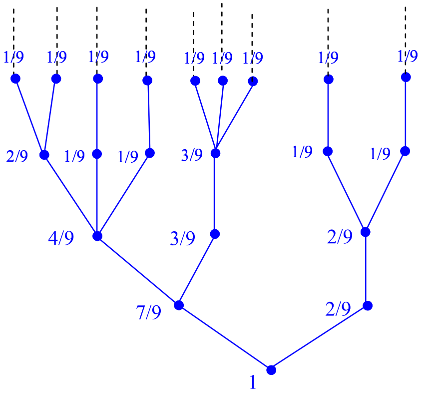

Multifractal analysis

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Let $\gamma(\beta) = -\log \mathbb{E}[N^{1+\beta}]$. Then

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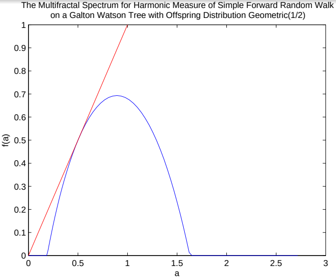

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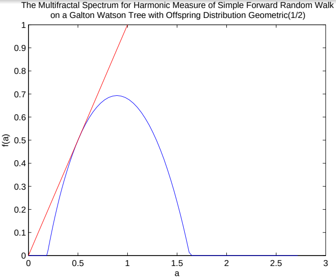

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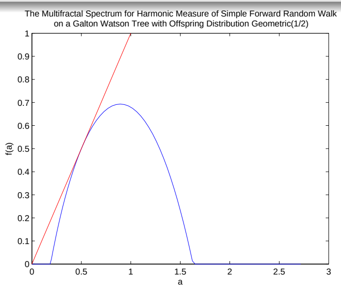

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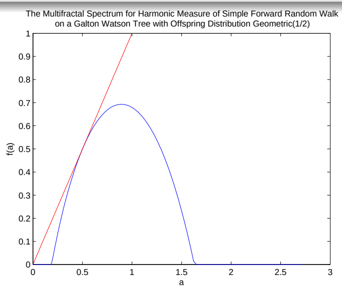

The branching measure μ

We assume that $\mathbb{P}\{N = 1\} > 0$.

Theorem 2 (Shieh + Taylor 2002, M + Shieh 2004)

Let $a = \log \mathbb{E}N$ and $\tau = -\log \mathbb{P}\{N = 1\}/a$. Then, almost surely,

$$\underline{\dim}_\mu(x) = a \quad \text{for all } x \in \partial T,$$

and, for all $a \le \theta \le a(1 + \frac{1}{\tau})$,

$$\dim\left\{x \in \partial T : \overline{\dim}_\mu(x) \ge \theta\right\} = \frac{1}{\theta}\, a^2(1+\tau) - a\tau.$$

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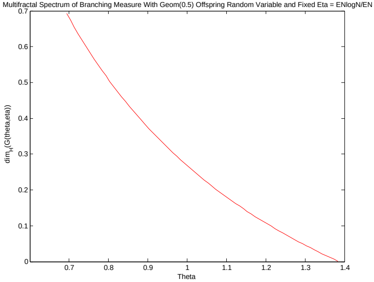

Branching and visibility measure

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Other references for simultaneous (or mixed) multifractal spectra:
Barreira and Schmeling 2000, Barreira and Saussol 2001, Olsen 2003.

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Theorem 3 (Kinnison + M 2009)

Let $a < \theta \leq a(1 + \frac{1}{\tau})$. Then, almost surely, for all $x \in \partial T$ we have that

$$\overline{\dim}_\mu(x) \geq \theta \quad \text{implies} \quad \overline{\dim}_\nu(x) \geq \frac{\theta}{a} \underline{\dim}_\nu(x),$$

and therefore the local dimension $\dim_\nu(x)$ never exists.

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Theorem 4 (Kinnison + M 2009)

Let $a \le \theta \le a(1+\frac{1}{\tau})$.

- If $\eta \le \eta_{\rm typ}$ then

$$\dim\big\{x \in \partial T : \overline{\dim}_\mu(x) \ge \theta, \overline{\dim}_\nu(x) \le \eta\big\} = \frac{1}{\theta} a^2\tau - a\tau + \frac{a}{\theta}\inf_\beta\big\{-\beta\eta - \gamma(\beta)\big\}.$$

- If $\eta \ge \frac{a}{\theta}\eta_{\rm typ}$ then

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Slices through the spectrum

Left branch: If $\eta \le \eta_{\mathrm{typ}}$, then

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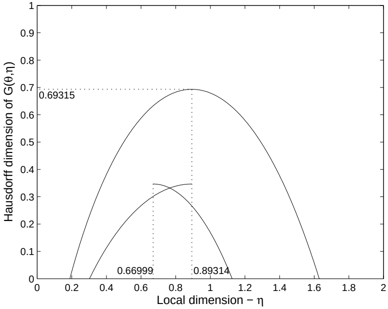
Right branch: If $\eta \ge \frac{a}{\theta}\, \eta_{typ}$, then

$$\dim\big\{x \in \partial T : \overline{\dim}_\mu(x) \ge \theta, \underline{\dim}_\nu(x) \ge \eta\big\} = \frac{1}{\theta} a^2 \tau - a\tau + \frac{a}{\theta} \inf_\beta \big\{ -\beta\eta \frac{\theta}{a} - \gamma(\beta)\big\}.$$



The geometry behind Theorem 2

We first look at the mulitfractal spectrum for the branching measure μ,

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The best strategy to make $\mu\{x \ni v\}$ small is to produce only one offspring for the initial k generations following v.

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The best strategy to make $\mu\{x \ni v\}$ small is to produce only one offspring for the initial k generations following v. This leads to

$$\frac{\log \mu\{x \ni v\}}{-m} \approx a + \frac{k}{m} a.$$

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are essentally those with, for infinitely many m, there is

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- and each independently has a chance of

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A refinement of this argument (based on percolation) yields the Hausdorff dimension of the set of rays with $\overline{\dim}_\mu(x) \geq \theta$.

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Suppose that we have a ray $x = (v_0, v_1, \ldots)$ such that we have

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Hence for a ray with $\overline{\dim}_\mu(x) \geq \theta$ necessarily

$\overline{\dim}_\nu(x)$ and $\underline{\dim}_\nu(x)$ differ by a factor of at least $\frac{\theta}{a}$

and therefore the local dimension cannot exist for such a ray.

The geometry behind Theorem 4

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The probability that a fixed vertex v is the first one in such a string is

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For m large, by the dimension spectrum for the visibility measure there are about

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$$\nu\{x \ni v_i\} \approx e^{-i\eta} \text{ for all } 1 \leq i \leq m.$$

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By the law of large numbers we therefore expect that we have about

$$\exp\Big\{m\big(- \tau(\theta - a) + \inf_{\beta}\{-\eta\beta - \gamma(\beta)\}\big)\Big\}$$

vertices in generation m satisfying these two constraints.

The geometry behind Theorem 4

First look at $\eta \leq \eta_{\text{typ}}$. If a vertex v in generation m satisfies the two constraints, then its descendants satisfy the required lower bound

$$\nu\{x \ni v_i\} \approx e^{-m\eta} \geq e^{-i\eta} \text{ for } i = m+1, \ldots, \tfrac{\theta}{a}m.$$

Therefore in generation $\frac{\theta}{a}m$ we expect about

$$\exp\{m(-\tau(\theta - a) + \inf_{\beta}\{-\eta\beta - \gamma(\beta)\})\}$$

vertices satisfying the constraint on the branching measure for the radius e^{-m} and the constraint on the visibility measure for all radii $1 \geq r \geq e^{-m\frac{\theta}{a}}$.

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Next look at $\eta \ge \eta_{\rm typ}^{-}$. If a vertex v in generation m satisfies the two constraints, then

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It is therefore necessary that the vertex v satisfies the stronger condition

$$\nu\{x \ni v_i\} \approx e^{-i\frac{\theta}{a}\eta} \text{ for } 1 \leq i \leq m.$$

to ensure that all descendants v_i in generations $i = m+1, \ldots, \frac{\theta}{a} m$ still satisfy the required upper bound.

Reference

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Available at `http://people.bath.ac.uk/maspm`