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Random colourings of aperiodic graphs: Ergodic and spectral properties

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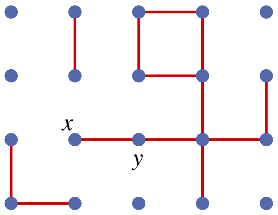
(joint work with Christoph Richard — U Erlangen)

1 Motivation: Laplacian on percolation graphs in $\mathbb{Z}^d$



random graph with realisations
$G_\omega = (\mathbb{Z}^d, \mathcal{E}_\omega)$

- $\{x, y\} \in \mathcal{E}_\omega$ with probability $p \in [0, 1]$, if $|x - y| = 1$, and zero otherwise
- different edges independently and identically distributed

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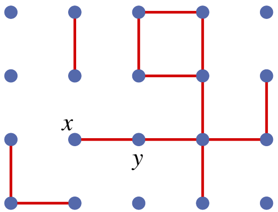

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Laplacian:

$$(\Delta_\omega \varphi)(x) := \sum_{y:\{x,y\}\in\mathcal{E}_\omega} [\varphi(x) - \varphi(y)]$$

$\forall\, x \in \mathbb{Z}^d \quad \forall\, \varphi \in \ell^2(\mathbb{Z}^d)$

- a.s. self-adjoint, ergodic w.r.t. $\mathbb{Z}^d$-translations
- $\operatorname{spec} \Delta_\omega = [0, 4d]$ a.s.
- integrated density of states:

$$N(E) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{\#\text{ eigenvalues of } \Delta_{\omega,\Lambda} \text{ not exceeding } E}{|\Lambda|}$$

$$= \int \mathbb{P}(d\omega') \, \langle \delta_0, \chi_{]-\infty,E]}(\Delta_{\omega'})\delta_0 \rangle$$

"number density of eigenvalues not exceeding E"

Theorem

(a) [Kirsch / Müller 06]
If $p < p_c$ then

$$\lim_{E \downarrow 0} \frac{\ln |\ln[N(E) - N(0)]|}{\ln E} = -\frac{1}{2}$$

(Lifshits tail)

(b) [Müller / Stollmann 07]
If $p > p_c$ then

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Idea of proof for (a):

linear clusters dominate!

Upper bound:

smallest positive eigenvalue for cluster containing the origin ↓

$$N(E) - N(0) \le \mathbb{P}\{E_1(\mathcal{C}_\omega) \le E\}$$

$$\le \mathbb{P}\{|\mathcal{C}_\omega|^{-2} \le E\}$$

Cheeger inequality: $E_1(\mathcal{C}_\omega) \ge |\mathcal{C}_\omega|^{-2}$ ↑

$$\le \exp\{-\text{const.}\ E^{-1/2}\}$$

↗ exponential decay of cluster-size distribution for $p < p_c$

Lower bound:
Only keep contributions from linear clusters in $\mathbb{P}$

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This talk: Replace underlying lattice $\mathbb{Z}^d$ by aperiodic graph

2 Ergodic theorem for randomly coloured point sets

Point space M: σ-compact, locally compact complete metric space

Uniformly discrete points sets in M of radius $r > 0$

$$\mathcal{P}_r(M) := \{P \subset M : |P \cap B_r(m)| \leq 1 \quad \forall\, m \in M\}$$

Local rubber metric on $\mathcal{P}_r(M)$:

$$\mathrm{dist}(P, Q) := \min\left\{\frac{1}{\sqrt{2}}, \inf\left(\varepsilon > 0 : P \cap B_{1/\varepsilon} \subset (Q)_\varepsilon \text{ and } Q \cap B_{1/\varepsilon} \subset (P)_\varepsilon\right)\right\}$$

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Translation group T: locally compact, Abelian, 2nd countable with continuous action on M

$\Longrightarrow$ acts continuously on $\mathcal{P}_r(M)$ by $x + P := \{x + p : p \in P\}$, where $x \in T$

Translation orbit of $\mathcal{P} \subset \mathcal{P}_r(M)$: $X_{\mathcal{P}} := \overline{\{x + P : x \in T, P \in \mathcal{P}\}}$ is compact!

$\Longrightarrow$ Birkhoff Ergodic Theorem available for top. dyn. system $(X_{\mathcal{P}}, T)$

colour space $\mathbb{A}$: σ-compact, locally compact complete metric space

$\Longrightarrow$ $\hat{M} := M \times \mathbb{A}$ is a point space $\Longrightarrow$ $\mathcal{P}_r(\hat{M})$ compact

translation: $x + \hat{P} := \{(x+p, a) \in \hat{M} : (p, a) \in \hat{P}\}$ for $\hat{P} \in \mathcal{P}_r(\hat{M})$, $x \in T$

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Definition. For $P \in \mathcal{P}_r(M)$ let $\Omega_P := \times_{p \in P} \mathbb{A}$

coloured point set $P_\omega := \{(p, \omega(p)) : p \in P\} \subset \widehat{M}$ for $\omega \in \Omega_P$

coloured translation orbit of $\mathcal{P} \subset \mathcal{P}_r(M)$: $\widehat{X}_{\mathcal{P}} := \{P_\omega : P \in X_{\mathcal{P}}, \omega \in \Omega_P\}$

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$\Longrightarrow$ $\quad \widehat{X}_{\mathcal{P}}$ compact

$\Longrightarrow$ $\quad$ Birkhoff Ergodic Theorem available for top. dyn. system $(\widehat{X}_{\mathcal{P}}, T)$

Random colour assignment to $P \in \mathcal{P}_r(M)$:

i.i.d. colours (for simpl.): $\quad \mathbb{P}_P := \times_{p \in P} \mathbb{P} \quad$ (on Ω_P; $\mathbb{P}$ a prob. meas. on $\mathbb{A}$)

Ergodic Theorem.

Let $\mathcal{P} \subseteq \mathcal{P}_r(M)$, fix an ergodic prob. meas. μ on $X_{\mathcal{P}}$.

Assume metric on M is T-invariant and T acts properly on M.

Then $\exists_1$ ergodic prob. meas. $\hat{\mu}$ on $\hat{X}_{\mathcal{P}}$ such that $\forall\, f \in L^1(\hat{X}_{\mathcal{P}}, \hat{\mu})$

(i) $$\int_{\hat{X}_{\mathcal{P}}} d\hat{\mu}(P_\omega)\, f(P_\omega) = \int_{X_{\mathcal{P}}} d\mu(P) \int_{\Omega_P} d\mathbb{P}_P(\omega)\, f(P_\omega)$$

(ii) $\forall$ tempered Følner sequences $(D_n)_{n\in\mathbb{N}} \subset T$

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(D_n)} \int_{D_n} dx\, f(x + \tilde{P}_{\tilde{\omega}}) = \int_{\hat{X}_{\mathcal{P}}} d\hat{\mu}(P_\omega)\, f(P_\omega)$$

for $\hat{\mu}$-a.a. $\tilde{P}_{\tilde{\omega}}$, in fact for μ-a.a. $\tilde{P} \in X_{\mathcal{P}}$ and for $\mathbb{P}_P$-a.a. $\tilde{\omega} \in \Omega_{\tilde{P}}$.

If $X_{\mathcal{P}}$ is uniquely ergodic and $f \in C(\hat{X}_{\mathcal{P}})$, then this holds even $\forall\, P \in X_{\mathcal{P}}$ and for $\mathbb{P}_P$-a.a. $\tilde{\omega} \in \Omega_{\tilde{P}}$.

History: Hof 93, Lee-Moody-Solomyak 02, Lenz-Veselic 08, Lenz 09

optimal for uniquely ergodic systems

M not neccessarily a group, T not neccessarily transitive on M

3 Randomly coloured graphs

Point space $\mathbb{W}$, $\quad M := \mathbb{W} \times \mathbb{W} / \sim \quad$ with equiv. rel. $(v, w) \sim (w, v)$

$\Longrightarrow \quad \mathcal{P}_r(M)$ compact.

Definition.

$G \in \mathcal{P}_r(M)$ is a graph (with uniformly discrete vertex set)

$:\Longleftrightarrow (v, w) \in G$ implies $(v, v) \in G$ and $(w, w) \in G$.

vertex set $\mathcal{V}_G := \{v \in \mathbb{W} : (v, v) \in G\}$ $\quad$ edge set $\mathcal{E}_G := G \setminus \{(v, v) : v \in \mathcal{V}_G\}$

Lemma.

$\mathcal{G}_r(M) := \{G \in \mathcal{P}_r(M) : \ G \text{ is a graph }\}$ is compact

continuous, proper translation action of T on $\mathbb{W}$ $\quad \Longrightarrow \quad$ so on M

$$x + (v, w) := (x + v, x + w)$$

random colouring and ergodic theorem as before...

4 Finite-range operators on randomly coloured graphs

Definition

Let G_ω be a coloured graph, H_{G_ω} be bounded and self-adjoint on $\ell^2(\mathcal{V}_G)$.

H_{G_ω} is covariant of finite range $R > 0$ $\quad :\Longleftrightarrow$

1. $\langle \delta_v, H_{G_\omega} \delta_w \rangle = 0, \quad$ if $\mathrm{dist}_{\mathbb{W}}(v,w) \geq R$
2. $\langle \delta_v, H_{G_\omega} \delta_w \rangle = \langle \delta_{x+v}, H_{G_\omega} \delta_{x+w} \rangle, \quad$ if for both $u = v$ and $u = w$:

$$x + \Big(G_\omega \cap B_R((u,u)) \Big) = G_\omega \cap B_R(x + (u,u))$$

Matrix elements depend only on local pattern!

Existence of the integrated density of states N, selfaveraging and non-randomness of the spectrum

Theorem

Fix $\mathcal{G} \in \mathcal{G}_r(M)$ and an ergodic prob. meas. $\hat{\mu}$ on $\hat{X}_{\mathcal{G}}$. Assume there is $R > 0$ s.t. $\forall\ G_\omega \in \hat{X}_{\mathcal{G}}$ $\quad H_\omega$ is covariant of finite range R. Then

(1) $\exists N : \mathbb{R} \to \mathbb{R}_{\geq 0}$, non-decreasing, s.t. for $\hat{\mu}$-a.e. $G_\omega \in \hat{X}_{\mathcal{G}}$

$$N(E) = \lim_{n\to\infty} \frac{1}{\mathrm{vol}(D_n)} \sum_{v \in \mathcal{V}_G \cap (D_n + v_0)} \langle \delta_v, \chi_{]-\infty,E]}(H_{G_\omega})\delta_v \rangle \qquad (*)$$

for all continuity points E of N. [Uniquely ergodic: $\forall\ G \in X_{\mathcal{G}}$ and $\mathbb{P}_G$-a.a. $\omega \in \Omega_G$]

(2) for $\hat{\mu}$-a.e. $G_\omega \in \hat{X}_{\mathcal{G}}$: $\quad \mathrm{spec}(H_{G_\omega}) = \mathrm{spec}_{ess}(H_{G_\omega}) = \mathrm{supp}(dN)$

[Uniquely ergodic and $\nu(P) > 0$ for all patterns P in G, $G \in X_{\mathcal{G}}$: $\forall\ G \in X_{\mathcal{G}}$ and $\mathbb{P}_G$-a.a. $\omega \in \Omega_G$]

Without colouring: (1) [Hof95, Lenz/Stollmann 05, Lenz/Peyerimhoff/Veselić 07]
(2) [Lenz/Stollmann 03]

5 Lifshits tails for the Laplacian

Assumptions:

1. Graph $G_0 \in \mathcal{G}_r(M)$ with
   - connected with vertex density >0
   - maximum bond length: $\ell_{max} := \sup\{\mathrm{dist}_{\mathbb{V}}(u,v) : (u,v) \in \mathcal{E}_{G_0}\} < \infty$
2. $\mathbb{A} = \{0,1\}$, $\mathbb{P}$ = Bernoulli(p)

   (bond probability $p \in]0,1[$)
3. $H_{G_\omega} = \Delta_\omega$ Laplacian for $G_\omega \in \hat{X}_{G_0}$

$$(\Delta_\omega \varphi)(v) := \sum_{u \in \mathcal{V}_G :\, \omega((u,v))=1} [\varphi(v) - \varphi(u)]$$

   self-adjoint, bounded,
   covariant of finite range
4. $N(E)$: int. dens. of states w.r.t. ergodic prob. measure $\hat{\mu}$ on $\hat{X}_{G_0}$

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Theorem

Assume (1) to (4). Let $p \in]0, p_c[$ s.t. with uniform constants D_p, λ_p

$$\mathbb{P}_G(|C_\omega^{(v)}| \geq n) \leq D_p\, e^{-\lambda_p n} \qquad (*)$$

$\forall\, n \in \mathbb{N}$ (exponential decay of cluster-size distribution). Then

$$\lim_{E \downarrow 0} \frac{\ln|\ln[N(E) - N(0)]|}{\ln E} = -\frac{1}{2}$$

Lemma

Let d_{max} be the maximal vertex degree of G_0 and assume $0 \leq p < \frac{1}{d_{max}-1}$. Then $(*)$ holds.

6 Outlook

- prove exponential decay of cluster-size distribution
  for all $p \in]0, p_c[$

  Menshikov Theorem? Aizenman / Barsky result?

- generalisation to random tilings (OK!)

- spectral asymptotics of N in the percolating phase?

  $\rightarrow$ further conditions needed on G_0 ?!

- Dirichlet Laplacians?