

Resistance analysis of infinite networks

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Definition (Electrical resistance network (G, c))

A *network* (G, c) is a simple, connected graph $G = \{G^0, G^1\}$ with vertices G^0 and edges G^1 .

The edges G^1 are determined by a weight function called *conductance*:

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$$x \sim y \text{ iff } 0 < c_{xy} < \infty, \quad \text{for } x, y \in G^0.$$

The conductance $c : G^0 \times G^0 \rightarrow [0, \infty)$ satisfies

$$c(x) := \sum_{y \sim x} c_{xy} < \infty, \text{ for all } x \in G^0, \text{ and}$$
$$c_{xy} = c_{yx} \text{ for all } x, y \in G^0.$$

Conductance is the reciprocal of the resistance. (Hence the name ERN.)

The energy form \mathcal{E}

Definition ((Dirichlet) energy of a function $u : G^0 \rightarrow \mathbb{C}$)

$$\mathcal{E}(u) := \frac{1}{2} \sum_{x,y \in G^0} c_{xy} |u(x) - u(y)|^2, \quad \text{dom } \mathcal{E} = \{u : \mathcal{E}(u) < \infty\}.$$

$c_{xy} = 0$ unless $x \sim y$; only pairs for which $x \sim y$.

“ $\frac{1}{2}$ ” indicates each edge is counted only once.

For $f : \mathbb{R} \rightarrow \mathbb{R}$, the continuous analogue is $\mathcal{E}(f) := \int |f'|^2 dx$.

Note: $\text{Ker } \mathcal{E} = \{\text{constant functions}\}$.

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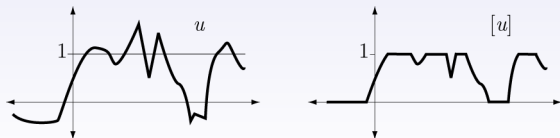
$$\mathcal{E}(u) := \frac{1}{2} \sum_{x,y \in G^0} c_{xy} |u(x) - u(y)|^2, \quad \text{dom } \mathcal{E} = \{u : \mathcal{E}(u) < \infty\}.$$

Definition (Energy form \mathcal{E} on functions $u, v \in \text{dom } \mathcal{E}$)

$$\mathcal{E}(u, v) := \frac{1}{2} \sum_{x,y \in G^0} c_{xy} (\bar{u}(x) - \bar{u}(y))(v(x) - v(y))$$

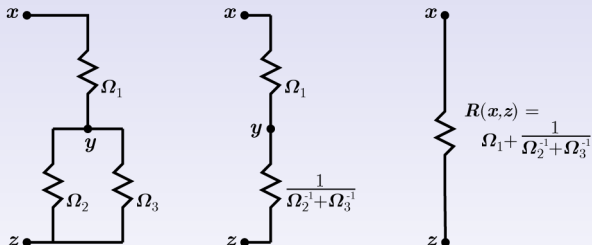
(Polarization) $\mathcal{E}(u, v) = \frac{1}{4} [\mathcal{E}(u + v) - \mathcal{E}(u - v)]$. (u, v \mathbb{R} -valued)

(Markov property) $\mathcal{E}([u]) \leq \mathcal{E}(u)$, where $[u]$ is any contraction of u .



Definition (Effective resistance)

The *effective resistance* $R(x, y)$ is the voltage drop between x and y when one unit of current is passed from x to y .



Series addition of resistors: $R = R_1 + R_2$.

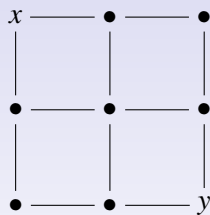
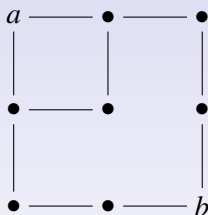
Parallel addition of resistors: $R = (R_1^{-1} + R_2^{-1})^{-1}$.

Theorem: Effective resistance $R(x, y)$ is a metric.

Resistance metric sees the topology of the graph

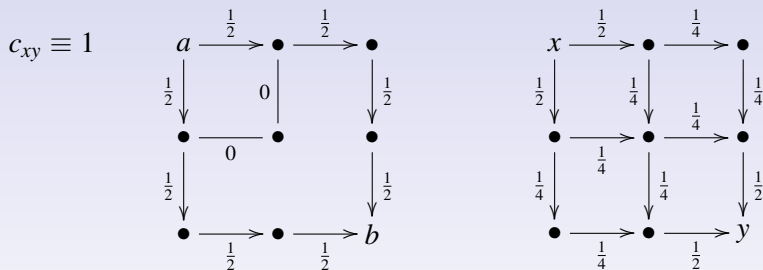
For shortest-path metric, $dist(a, b) = 4 = dist(x, y)$.

$$c_{xy} \equiv 1$$



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For shortest-path metric, $dist(a, b) = 4 = dist(x, y)$.



Points are R -closer when there are more paths between them:

$$R(a, b) = 2 > 1\frac{1}{2} = R(x, y).$$

$R(x, y)$ is closely related to Δ and the random walk

Let X_n be a RW started at x , i.e., $X_0 = x$.

Then the *probability of reaching b before a* is

$$u(x) = \frac{v(x)}{R(a, b)} = \mathbb{P}[\tau_b < \tau_a].$$

Here, u, v are the extremizers from

$$\begin{aligned} R(a, b) &= 1 / \min\{\mathcal{E}(u) : |u(a) - u(b)| = 1\} \\ &= \min\{|v(a) - v(b)| : \Delta v = \delta_a - \delta_b\} \\ &= \min\{\mathcal{E}(v) : \Delta v = \delta_a - \delta_b\} \\ &= \min\{\kappa \geq 0 : |v(a) - v(b)|^2 \leq \kappa \mathcal{E}(v)\} \\ &= \max\{|v(a) - v(b)|^2 : \mathcal{E}(v) \leq 1\} \end{aligned}$$

Again, the RW has $p(x, y) = \frac{c_{xy}}{c(x)}$, where $c(x) := \sum_{y \sim x} c_{xy}$.

Two roads to the energy space $\mathcal{H}_{\mathcal{E}}$

(1) Direct construction.

Define $\mathcal{H}_{\mathcal{E}} := \text{dom } \mathcal{E} / \mathbb{C}\mathbf{1}$, $\langle u, v \rangle_{\mathcal{E}} := \mathcal{E}(u, v)$.

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(2) von Neumann embedding.

The metric $R = R^F$ is negative semidefinite.

Therefore, (G, R) embeds into a Hilbert space (von Neumann):

$$x \mapsto v_x \text{ with } R(x, y) = \|v_x - v_y\|_{\mathcal{E}}^2.$$

Up to unitary equivalence, this Hilbert space is $\mathcal{H}_{\mathcal{E}}$.

Two roads to the energy kernel

- (1) Fix $o \in G^0$ (origin). Define $L_x : \mathcal{H}_{\mathcal{E}} \rightarrow \mathbb{C}$ by $L_x u := u(x) - u(o)$.
 L_x is continuous on $\mathcal{H}_{\mathcal{E}}$, so $L_x u = \langle v_x, u \rangle_{\mathcal{E}}$ for some $v_x \in \mathcal{H}_{\mathcal{E}}$.
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- (2) von Neumann embedding: $x \mapsto v_x$ with $R(x, y) = \|v_x - v_y\|_{\mathcal{E}}^2$.

Both yield the same set of embedded vertices $\{v_x\}_{x \in G^0}$.

Definition: The collection $\{v_x\}_{x \in G^0}$ is the *energy kernel*.

Theorem: $\{v_x\}_{x \in G^0}$ is a *reproducing kernel*:

$$\langle v_x, u \rangle_{\mathcal{E}} = u(x) - u(o) \text{ for all } u \in \mathcal{H}_{\mathcal{E}}.$$

Choosing representatives with $u(o) = 0$ gives the (discrete) Kuramochi kernel.

Definition ((Network) Laplacian Δ)

A linear difference operator; weighted average of neighbouring values.

$$(\Delta v)(x) := \sum_{y \sim x} c_{xy}(v(x) - v(y)).$$

If the operator c is multiplication by $c(x) := \sum_{y \sim x} c_{xy}$, then

$$\Delta = c(\mathbb{I} - \mathbf{P}),$$

where \mathbf{P} is the *(probabilistic) transition operator*

$$(\mathbf{P}v)(x) := \frac{1}{c(x)} \sum_{y \sim x} c_{xy}v(y).$$

(stochastically renormalized adjacency matrix)

The structure of $\mathcal{H}_\mathcal{E}$

Definition: $u \in \mathcal{H}_\mathcal{E}$ is a *dipole* iff $\Delta u = \delta_x - \delta_y$ for some $x, y \in G^0$.

Definition: $w \in \mathcal{H}_\mathcal{E}$ is a *monopole* iff $\Delta w = \delta_x$ for some $x \in G^0$.

Definition: $h \in \mathcal{H}_\mathcal{E}$ is *harmonic* iff $\Delta h(x) = 0$ for each $x \in G^0$.

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Theorem: v_x is a dipole: $\Delta v_x = \delta_x - \delta_o$.

Theorem: \mathcal{H}_E contains monopoles iff the network is transient.

Theorem: If \mathcal{H}_E contains nonconstant harmonics, the network is transient.

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Theorem: $\mathcal{H}_{\mathcal{E}} = \mathcal{F}in \oplus \mathcal{H}arm$.

$\mathcal{H}arm := \{h \in \mathcal{H}_{\mathcal{E}} : \Delta h(x) = 0, \forall x \in G^0\}$, and

$\mathcal{F}in := [\{f \in \mathcal{H}_{\mathcal{E}} : f(x) = k, \text{ for all but finitely many } x \in G^0\}]_{\mathcal{E}}$.

Let w_o be the unique energy-minimizing monopole at the origin.
 Define $w_x^v := v_x + w_o$ and $w_x^f := f_x + w_o$, where $f_x = P_{\mathcal{F}in}v_x$.

Definition: Let $\mathcal{M} := \text{span}\{v_x, w_x^v, w_x^f\}_{x \in G^0}$.

\mathcal{M} always contains: $\text{span}\{v_x\}$, $\text{span}\{f_x\}$, $\text{span}\{h_x\}$, $h_x = P_{\mathcal{H}arm}v_x$.

Let w_o be the unique energy-minimizing monopole at the origin. Define $w_x^v := v_x + w_o$ and $w_x^f := f_x + w_o$, where $f_x = P_{\mathcal{F}_m} v_x$.

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Theorem (Discrete Gauss-Green Formula)

For $u \in \mathcal{H}_{\mathcal{E}}$ and $v \in \mathcal{M}$, $\langle u, v \rangle_{\mathcal{E}} = \sum_{G^0} u \Delta v + \sum_{\text{bd } G} u \frac{\partial v}{\partial \mathbf{n}}$.

$$\int_U \nabla \varphi \cdot \nabla \psi \, dV = - \int_U \varphi \Delta \psi \, dV + \int_{\partial U} \varphi \frac{\partial}{\partial \mathbf{n}} \psi \, dS$$

$$\sum_{\text{bd } G} u \frac{\partial v}{\partial \mathbf{n}} := \lim_{k \rightarrow \infty} \sum_{x \in \text{bd } G_k} u(x) \frac{\partial v}{\partial \mathbf{n}}(x).$$

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Let $G_k \subseteq G_{k+1}$, $G = \bigcup G_k$, as before.

$$\text{bd } G_k := \{x \in G_k : \exists y \in G_k^{\mathbf{c}}, y \sim x\}$$

$$\frac{\partial v}{\partial \mathbf{n}}(x) := \sum_{y \in G_k} c_{xy}(v(x) - v(y)), \quad x \in \text{bd } G_k$$

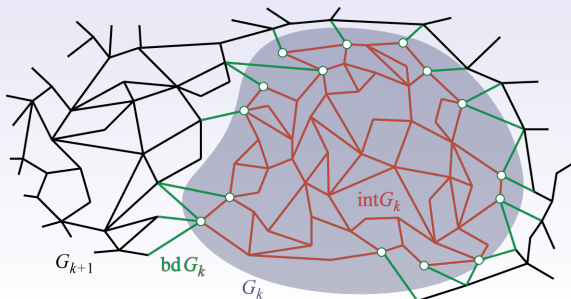
Think: $\frac{\partial v}{\partial \mathbf{n}}(x) = \Delta|_{G_k}(x)$.

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Corollaries of the discrete Gauss-Green Formula

Theorem (Discrete Gauss-Green Formula)

For $u \in \mathcal{H}_\varepsilon$ and $v \in \mathcal{M}$, $\langle u, v \rangle_\varepsilon = \sum_{G^0} u \Delta v + \sum_{\text{bd } G} u \frac{\partial v}{\partial \mathbf{n}}$.

Theorem: The network is transient if and only if the boundary term is nonvanishing.

I.e., nonzero boundary sum when applied to $\langle u, v \rangle_\varepsilon$, for every representative of u except the one specified by $u(x) = \langle u, w_x^v \rangle_\varepsilon$.

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Theorem: The network is transient if and only if $(\text{ran } \Delta_{\mathcal{M}}^*)^{cl} = \mathcal{F}in$.

$\Delta_{\mathcal{M}}$ is the closure of Δ when taken to have dense domain \mathcal{M} .

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Theorem [Boundary representation]: For all $u \in \mathcal{H}_{\text{arm}}$,

$$u(x) = \sum_{\text{bd } G} u \frac{\partial h_x}{\partial \mathfrak{n}} + u(o).$$

Proof: Note that $u(x) - u(o) = \langle v_x, u \rangle_\varepsilon = \overline{\langle u, h_x \rangle_\varepsilon} = \sum_{\text{bd } G} u \frac{\partial h_x}{\partial \mathfrak{n}}$.

$$u(x) = \sum_{\text{bd } G} u \frac{\partial h_x}{\partial \mathfrak{n}} + u(o) \quad \text{vs.} \quad u(x) = \int_{\partial\Omega} u(y)k(x, dy), \quad y \in \partial\Omega.$$

Goals:

- A measure space $\text{bd } G$ and a measure \mathbb{P} on it.
- An extension of $u, h_x \in \mathcal{H}_{\text{arm}}$ to elements $\xi \in \text{bd } G$.
- A kernel $\mathbb{k}(x, d\xi) := h_x(\xi)d\mathbb{P}(\xi)$ on $G^0 \times \text{bd } G$.
- An integral representation $u(x) = \int_{\text{bd } G} u(\xi)\mathbb{k}(x, d\xi) + u(o)$.
- A concrete realization of $\xi \in \text{bd } G$.

Problem: \mathcal{H}_E is too small to support \mathbb{P} .

Theorem (Nelson): If μ is a σ -finite measure on a Hilbert space H , then $\mu H = 0$.

Problem: $\mathcal{H}_\mathcal{E}$ is too small to support \mathbb{P} .

Theorem (Nelson): If μ is a σ -finite measure on a Hilbert space H , then $\mu H = 0$.

Solution: construct a **Gel'fand triple** for $\mathcal{H}_\mathcal{E}$.

$$\mathcal{S} \subseteq \mathcal{H}_\mathcal{E} \subseteq \mathcal{S}'.$$

- \mathcal{S} is dense in $\mathcal{H}_\mathcal{E}$ with respect to \mathcal{E} .
- \mathcal{S} has another, strictly finer, “test function” topology.
- \mathcal{S}' is the dual of \mathcal{S} with respect to the test function topology.

Think: $\mathcal{S} = \{\text{test functions}\}$ and $\mathcal{S}' = \{\text{distributions}\}$.

The boundary will be some suitable subspace of \mathcal{S}' .

Constructing the space of test functions

(1) Take a self-adjoint extension $\Delta_{\mathcal{M}}$.

$\Delta_{\mathcal{M}}$ is Hermitian and commutes with conjugation, so it has. (von Neumann)

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(2) Define $\mathcal{S} := \text{dom}(\Delta_{\mathcal{M}}^{\infty}) := \bigcap_{p=1}^{\infty} \text{dom}(\Delta_{\mathcal{M}}^p)$.

Here $\Delta_{\mathcal{M}}^p := \underbrace{(\Delta_{\mathcal{M}} \Delta_{\mathcal{M}} \dots \Delta_{\mathcal{M}})}_p$, so $\mathcal{S} \subseteq \text{dom} \Delta_{\mathcal{M}}$ and $\Delta_{\mathcal{M}}(\mathcal{S}) \subseteq \mathcal{S}$.

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(3) **Theorem:** \mathcal{S} is a dense analytic subspace of $\mathcal{H}_\mathcal{E}$ (w.r. \mathcal{E}) and \mathcal{S} is also a Fréchet space with seminorms $\|u\|_p := \|\Delta_{\mathcal{M}}^p u\|_\mathcal{E}$.

(Approximate using $E_n u := \int_{1/n}^n E(dt)u$.)

(4) $\mathcal{S} \subseteq \mathcal{H}_\mathcal{E}$, so $\mathcal{H}'_\mathcal{E} \subseteq \mathcal{S}'$, and then $\mathcal{H}_\mathcal{E} = \mathcal{H}'_\mathcal{E}$ gives a Gel'fand triple $\mathcal{S} \subseteq \mathcal{H}_\mathcal{E} \subseteq \mathcal{S}'$.

$\mathcal{S} \subseteq \mathcal{H}_\mathcal{E} \subseteq \mathcal{S}'$ is a Gel'fand triple for $\mathcal{H}_\mathcal{E}$

(1) The energy form extends to a pairing on $\mathcal{S} \times \mathcal{S}'$:

$$\langle u, \xi \rangle_{\overline{\mathcal{E}}} = \langle \Delta_{\mathcal{M}}^p u, \Delta_{\mathcal{M}}^{-p} \xi \rangle_{\mathcal{E}} = \lim_{n \rightarrow \infty} \xi(E_n u).$$

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(2) The energy form extends to a pairing on $\mathcal{H}_\mathcal{E} \times \mathcal{S}'$.

(3) **Theorem.** The Wiener transform

$$\mathcal{W} : \mathcal{H}_\mathcal{E} \rightarrow L^2(\mathcal{S}'_G, \mathbb{P}), \quad \mathcal{W} : u \mapsto \tilde{u}(\xi) := \langle u, \cdot \rangle_{\mathcal{W}}$$

gives an isometric embedding of $\mathcal{H}_\mathcal{E}$ into an L^2 space:

$$\langle u, v \rangle_{\mathcal{E}} = \int_{\mathcal{S}'} \tilde{u} \tilde{v} d\mathbb{P}.$$

The proof uses:

Bochner: $R^F(x, y) = \|v_x - v_y\|_{\mathcal{E}}^2$ is negative semidefinite, so $e^{-\frac{1}{2}\|u-v\|_{\mathcal{E}}^2}$ is positive definite.

Minlos: {pos. def. fns on \mathcal{S} } \leftrightarrow {Radon prob. meas. on \mathcal{S}' }.

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(4) **Theorem.** For $u \in \mathcal{H}_{\text{arm}}$ and $h_x = P_{\mathcal{H}_{\text{arm}}} v_x$,

$$u(x) = \int_{\mathcal{S}'} u(\xi) h_x(\xi) d\mathbb{P}(\xi) + u(o).$$

Boundary integral representation of $u \in \mathcal{H}_{\text{arm}}$

Theorem. For $u \in \mathcal{H}_{\text{arm}}$ and $h_x = P_{\mathcal{H}_{\text{arm}}} v_x$,

$$u(x) = \int_{\mathcal{S}'} u(\xi) h_x(\xi) d\mathbb{P}(\xi) + u(o).$$

From the Wiener isometry:

$$L^2(\mathcal{S}', \mathbb{P}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathcal{E}}^{\otimes n} = \mathbb{C}\mathbf{1} \oplus \mathcal{H}_{\mathcal{E}} \oplus \mathcal{H}_{\mathcal{E}}^2 \oplus \dots$$

$\mathcal{H}_{\mathcal{E}}^{\otimes 0} := \mathbb{C}\mathbf{1}$ for a unit “vacuum” vector $\mathbf{1}$.

$\mathcal{H}_{\mathcal{E}}^{\otimes n}$ is the n -fold symmetric tensor product of $\mathcal{H}_{\mathcal{E}}$ with itself.

$u \mapsto \langle u, \cdot \rangle \in \mathcal{H}_{\mathcal{E}}^1$, $(u, v) \mapsto \langle u, \cdot \rangle \langle v, \cdot \rangle \in \mathcal{H}_{\mathcal{E}}^2$, etc.

Observe that $\mathbf{1}$ is orthogonal to \mathcal{F}_{in} and \mathcal{H}_{arm} , but is not the zero element of $L^2(\mathcal{S}'_G, \mathbb{P})$.

Resistance analysis of infinite networks

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The kernel $\mathbb{k}(x, d\xi) = (\mathbf{1} + h_x)d\mathbb{P}$

Since $u(x) = \int_{\mathcal{S}'} u(\xi)h_x(\xi) d\mathbb{P}(\xi) + u(o)$, the obvious choice for the kernel is $h_x d\mathbb{P}$.

$k(x, \cdot) = h_x d\mathbb{P}$ is supported on $\mathcal{S}'/\mathcal{F}in$:

since h_x is harmonic, $h_x(\xi) = \langle h_x, \xi \rangle_{\mathcal{W}} = 0$ for $\varphi \in \mathcal{F}in$.

But then $\int_{\mathcal{S}'} k(x, \cdot) d\mathbb{P} = \int_{\mathcal{S}'} \mathbf{1}h_x d\mathbb{P} = \langle \mathbf{1}, h_x \rangle_{\mathcal{E}} = 0$.

One expects $\int_{\mathcal{S}'} k(x, \cdot) d\mathbb{P} = 1$.

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Now $\int_{S'} \mathbb{k}(x, d\xi) d\mathbb{P} = \int_{S'} \mathbf{1} d\mathbb{P} + \int_{S'} h_x d\mathbb{P} = 1$.

It follows that $h_x \geq -1$ \mathbb{P} -a.e. on S' .

Also, $\mathbb{k}(x, \cdot) \ll \mathbb{P}$ with Radon-Nikodym derivative $\frac{d\mathbb{k}_x}{d\mathbb{P}} = \mathbf{1} + h_x$.

The boundary $\text{bd } G$

A path is a sequence of vertices $\gamma = (x_0, x_1, \dots)$ with $x_i \sim x_{i-1}$.

Define $\gamma \simeq \gamma'$ iff $\lim_{n \rightarrow \infty} (h(x_n) - h(x'_n)) = 0$ for every $h \in \mathcal{Harm}$.

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Alaoglu's theorem gives a weak- \star limit, so for any $u \in \mathcal{Harm}$,

$$u(x_n) = \int_{S'} u(\mathbf{1} + h_x) d\mathbb{P} \xrightarrow{n \rightarrow \infty} \int_{S'} u d\nu_\beta := u(\beta).$$

So $\text{bd } G$ is the set of all equivalence classes of infinite paths in G , under this equivalence relation.

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Compare $\mathbb{k}(x_n, d\mathbb{P})$ to an approximate identity in Fourier analysis:

$$\int_{\mathcal{S}'} \mathbb{k}(x_n, d\mathbb{P}) = 1 \text{ for each } n, \text{ and}$$

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Intuition: on any finite subset G_k , define a probability measure μ_x on $\text{bd } G_k$ by

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Consider Brownian motion on a disk with such an exit measure.



Definition (Effective resistance)

The *effective resistance* $R(x, y)$ is the voltage drop between x and y when one unit of current is passed from x to y .

$$R^F(x, y) = (v_x - v_y)(x) - (v_x - v_y)(y) \quad (1a)$$

$$= \mathcal{E}(v_x - v_y) \quad (1b)$$

$$= \min\{D(I) : \operatorname{div} I = \delta_x - \delta_y \text{ and } I = \sum \xi_\gamma \chi_\gamma\} \quad (1c)$$

$$= (\min\{\mathcal{E}(v) : v(x) = 1, v(y) = 0\})^{-1} + \mathcal{E}(P_{\mathcal{H}arm}(v_x - v_y)) \quad (1d)$$

$$= \inf\{\kappa \geq 0 : |v(x) - v(y)|^2 \leq \kappa \mathcal{E}(v), \forall v \in \operatorname{dom} \mathcal{E}\} \quad (1e)$$

$$= \sup\{|v(x) - v(y)|^2 : \mathcal{E}(v) \leq 1, \forall v \in \operatorname{dom} \mathcal{E}\} \quad (1f)$$

Let $\{G_k\}_{k=1}^\infty$ be an *exhaustion* of G and define $R_{G_k}(a, b)$ as above, except extremize over $u, v : G^0 \rightarrow \mathbb{C}$. Then define $R^F(x, y) := \lim_{k \rightarrow \infty} R_{G_k}(x, y)$.

Theorem: $x \mapsto v_x$ is an isometric embedding of (G, R^F) into $\mathcal{H}_\mathcal{E}$:

$$R^F(x, y) = \|v_x - v_y\|_\mathcal{E}^2.$$