Advanced and algorithmic graph theory

Summer term 2017

Exercise sheet 1

Exercises for the exercise session on 20/03/2017

Problem 1.1. Let G be a weighted graph. For $i = 1, 2, 3, 4$, denote by T_i a spanning tree that is generated by Algorithm i from the lecture. Reminder:

Alg. 1: Recursively add edges of smallest weight that do not create cycles.

Alg. 2: Recursively delete edges of largest weight that do not separate.

Alg. 3: Start at a vertex v and recursively connect new vertices to the current tree with an edge of smallest weight.

Alg. 4: (Only applicable if all edge weights are distinct) Perform Algorithm 3 in parallel from each vertex.

Prove that each spanning tree T_i has minimum total weight among all spanning trees of G. If all weights are distinct, prove that there is a unique spanning tree of minimum weight.
Hint. For $i = 1, 2, 3$, let T be a spanning tree of minimum total weight with as many edges in common with T_i as possible. For $i = 4$, let T be any spanning tree of minimum weight. In either case, prove that $T \neq T_i$ implies that there is a tree of smaller weight than T.

Problem 1.2. Let G be a graph with at least two vertices. Prove that

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

For every integer $k \geq 1$, construct graphs G_1, G_2 with

$$\kappa(G_1) = 1, \ \ \lambda(G_1) = \delta(G_1) = k,$$
$$\kappa(G_2) = \lambda(G_2) = 1, \ \ \delta(G_2) = k.$$

Problem 1.3. For a graph G, we define the *line graph* $L(G)$ to be the graph with vertex set $E(G)$ and an edge between e and f iff e and f share a vertex.
Prove that for every G with at least three vertices and one edge (why do we need these extra conditions?) we have

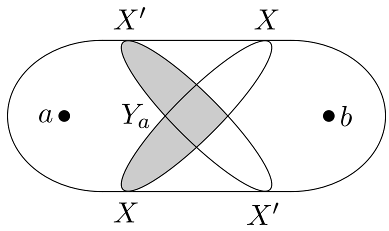
$$\kappa(L(G)) \geq \lambda(G).$$

Give an example of a connected G for which the two values are not the same.

Problem 1.4. Let G be a graph and let a, b be two distinct vertices of G. Suppose that each of the vertex sets $X, X' \subseteq V(G) \setminus \{a, b\}$ is an a-b separator. Denote by C_a and C_b the component of $G - X$ that contains a and b, respectively. Define C'_a and C'_b analogously for X'. Prove that the sets

$$Y_a := (X \cap C'_a) \cup (X \cap X') \cup (X' \cap C_a),$$
$$Y_b := (X \cap C'_b) \cup (X \cap X') \cup (X' \cap C_b)$$

are a-b separators as well.



Problem 1.5. Let G, a, b, X, X', Y_a, Y_b be as in Problem 1.4. Prove that X is a minimal a-b separator iff each vertex in X has neighbours in both C_a and C_b. If both X and X' are minimal a-b separators, are Y_a and Y_b minimal as well? Do Y_a and Y_b have smallest size (among all separators in $V \setminus \{a, b\}$) if both X and X' have?

Problem 1.6. Let G be k-connected, where $k \geq 2$. Prove that for each set of k vertices in G, there is a cycle in G that contains them all.