Advanced and algorithmic graph theory

Summer term 2017

Exercise sheet 2

Exercises for the exercise session on 03/04/2017

Problem 2.1. Prove Theorem 1.19. from the lecture: A graph G is 2-edge-connected iff there exists an orientation of the edges of G such that the resulting directed graph is strongly connected.

Problem 2.2. Let G be bipartite with sides A and B and let $A' \subseteq A$ and $B' \subseteq B$. Suppose that G contains matchings M_A and M_B that cover A' and B', respectively. Prove that G contains a matching that covers both A' and B'.

Problem 2.3. Let G be bipartite and let U be the set of vertices of degree $\Delta(G)$. Prove that G contains a matching that covers U. Deduce that in particular every r-regular bipartite graph with $r > 0$ has a perfect matching.

Problem 2.4. Let G be bipartite with sides A and B and suppose that each vertex v has a preference order $\geq_v$ on its set of neighbours. Let us call a stable matching M *A-optimal* if each vertex $a \in A$ is matched to its 'best' neighbour among all vertices that are possible as partners for a in a stable matching. (Formally: If $ab \in M$ and $ab' \in M'$ for some stable matching M', then $b \geq_a b'$.) We define *B-optimal*, *A-pessimal* (worst possible for A) and *B-pessimal* analogously.
Prove that a stable matching is A-optimal iff it is B-pessimal. Furthermore, prove that every bipartite G has a unique A-optimal matching.

Problem 2.5. In the algorithm Stable, if at some time $a \in A$ has proposed to $b \in B$ and b has not rejected the proposal (possibly only not rejected it *yet*), we call the proposal *open*. Consider the following variant of Stable. In each step, we choose a vertex $a \in A$ with no open proposals and let it propose to its 'best' remaining neighbour, or we choose a vertex $b \in B$ with at least two open proposals and let it reject all but its 'best' suitor. The algorithm ends once no such vertices in A and B exist. By the same arguments as for Stable, this variant produces a stable matching M.
Prove that M is A-optimal (and thus also B-pessimal), regardless of the choices that are made.
Hint. Consider the first time that some vertex $a \in A$ is refused by its 'best' choice b among its possible partners.

Problem 2.6. A graph is called *transitive* if for every two vertices u, v, there exists an automorphism (a bijective map $\varphi\colon V \to V$ for which $xy \in E$ iff $\varphi(x)\varphi(y) \in E$) that maps u to v.
Prove that every transitive graph with an even number of vertices has a perfect matching.