Advanced and algorithmic graph theory

Summer term 2017

Exercise sheet 3

Exercises for the exercise session on 08/05/2017

Problem 3.1. Let G be a 2-connected plane graph. Prove that each face of G is bounded by a cycle. Deduce that if G is 3-connected, then for each vertex x, the face of $G - x$ that contains x is bounded by a cycle (this was used in the proof of Lemma 3.7).

Problem 3.2. Let G, H be graphs.

(a) Prove that $G = TH$ implies $G = MH$, i.e. every topological minor is a minor.

(b) Prove that if $\Delta(H) \leq 3$ and $MH \subseteq G$, then $TH \subseteq G$.

(c) Prove that for every surface S, the set of forbidden subdivisions is finite.

Hint. Use the corresponding statement for minors.

Problem 3.3. A planar graph is called *outerplanar* if it has a drawing in which all vertices lie on the boundary of the outer face. Prove that a graph is outerplanar iff it contains neither K^4 nor $K_{2,3}$ as a minor.

Problem 3.4. Let G be a non-planar graph. Prove or disprove the following claims.

(a) If every proper subgraph of G is planar, there exists an edge e such that $G - e$ is maximally planar.

(b) If every proper minor of G is planar, there exists an edge e such that G/e is maximally planar.

Problem 3.5. Let G, H be graphs on the same vertex set such that G is maximally planar and H is obtained from G by adding one edge. Prove the following statements.

(a) H contains a TK^5.

(b) If H (and thus also G) has at least six vertices, then it contains a $TK_{3,3}$.

Problem 3.6. Let G be a graph that is drawn on the torus and consider the following properties.

(i) G is a maximal drawing (i.e. no edge can be added without creating a crossing);

(ii) G is maximally embeddable on the torus (i.e. for each $e \notin E(G)$, the *abstract* graph $G + e$ is not embeddable on the torus);

(iii) G is a triangulation;

(iv) $m = 3n$.

Prove that (i) follows from each of the other three properties. Vice versa, prove that there is no $k \in \mathbb{N}$ such that (i) implies (ii), (iii), or (iv) for $n \geq k$.
Hint. Find a 'small' example first and then try to extend it to arbitrary size.