Advanced and algorithmic graph theory

Summer term 2017

Exercise sheet 4

Exercises for the exercise session on 22/05/2017

Problem 4.1. Find all mistakes in the following "proof" of the four colour theorem.

Suppose, for contradiction, that the four colour theorem is false. Let v be a vertex of degree $d := \delta(G) \leq 5$ in a smallest non-4-colourable planar graph G. Fix a drawing of G and a 4-colouring c of $G - v$. Denote the neighbours of v by $x_1, \ldots, x_d$ in the order they lie around v in the drawing. Furthermore, set $G_{i,j} := G[c^{-1}(i) \cup c^{-1}(j)]$. Since G is not 4-colourable, we know that

$$\text{no 4-colouring of } G - v \text{ uses less than four colours for } N(v). \tag{1}$$

In particular, $d \geq 4$. W.l.o.g. $c(x_i) = i$ for $i = 1, 2, 3, 4$ and, if $d = 5$, $c(x_5) \in \{1, 2\}$.

Suppose first that $d = 4$. If there is no x_1-x_3 path in $G_{1,3}$, then we can recolour x_1 with colour 3 by exchanging the colours in the component of $G_{1,3}$ that contains x_1 and obtain a colouring c' that contradicts (1). Otherwise, we can recolour x_2 with colour 4 analogously.

Now suppose that $d = 5$ and $c(x_5) = 1$. If there is no x_3-$\{x_1, x_5\}$ path in $G_{1,3}$, we can recolour x_3 with colour 1. Otherwise, we can recolour x_2 with colour 4, again yielding a contradiction.

Finally, suppose that $d = 5$ and $c(x_5) = 2$. If there is no x_1-x_3 path in $G_{1,3}$ or no x_1-x_4 path in $G_{1,4}$, then we can recolour x_1 with colour 3 or 4, respectively. Otherwise, there is neither an x_2-x_4 path in $G_{2,4}$ nor an x_5-x_3 path in $G_{2,3}$. Thus we can recolour x_2 with colour 4 and x_5 with colour 3, again a contradiction to (1).

Problem 4.2. Prove that there exists an $\alpha > 0$ with the following property.
For every $k \in \mathbb{N}$, there are bipartite graphs G_k, H_k on at least k vertices such that

(a) $\Delta(G_k) \geq \alpha\,|V(G_k)|$ and $\Delta(H_k) \geq \alpha\,|V(H_k)|$,

(b) $\chi_{\mathrm{Gr}}(G_k, \sigma_0) = \Delta(G_k) + 1$ for some ordering σ_0 of $V(G_k)$, and

(c) $\chi_{\mathrm{Gr}}(H_k, \sigma) = 2$ for every ordering σ of $V(H_k)$.

Problem 4.3. Given a non-empty graph G and $k \in \mathbb{N}$, denote by $P_G(k)$ the number of colourings $c\colon V(G) \to \{1, \ldots, k\}$ (not every colour has to be used). Prove that

$$P_G(k) = k^n - mk^{n-1} + a_{n-2}k^{n-2} + \cdots + a_0\,.$$

(P_G is also called the *chromatic polynomial.*)
Hint. Induction on m.

Problem 4.4. Determine all graphs that satisfy $P_G(k) = k(k-1)^{n-1}$.

Problem 4.5. A graph G is called *colour-critical* if $\chi(G - v) = \chi(G) - 1$ for every vertex v. Prove that every colour-critical graph G with $n \geq 3$ is $(\chi(G) - 1)$-edge-connected.

Problem 4.6. Along the lines of the proof of Brooks' theorem, derive an algorithm that produces a $\Delta(G)$-colouring in time $O(m + n)$, provided that G is neither complete nor an odd cycle.