Advanced and algorithmic graph theory

Summer term 2017

Exercise sheet 6

Exercises for the exercise session on 29/06/2017

Problem 6.1. Pick one of the proofs of Dirac's theorem and show that it can also be applied to prove Ore's theorem. Based on the proof, describe an algorithm that finds a Hamilton cycle in a given graph G that satisfies the condition of Ore's theorem. What running time can you achieve? What running time is necessary to check whether G satisfies the condition of Dirac's or Ore's theorem, respectively?

Problem 6.2. Suppose that G satisfies $d(x)+d(y) \geq n$ for all $x \neq y$ with $xy \notin E(G)$. Prove that $\kappa(G) \geq \alpha(G)$.

(In other words, the Chvátal-Erdős theorem implies Ore's theorem.)

Problem 6.3. Recall that the square G^2 of a graph G has the same vertex set as G and an edge between v and w iff v and w have distance at most two in G.

(a) Prove that if G is k-connected, then G^2 is k-tough.

(b) Find a connected graph G on at least three vertices for which G^2 is not hamiltonian.

Note. The graph G^2 from (b) is a graph that is not hamiltonian, but 1-tough by (a).

Problem 6.4. Suppose that precisely two vertices x, y in a graph G have even degrees. Denote by $\mathcal{H}$ the (possibly empty) set of all Hamilton paths in G that start in x. Define an auxiliary graph H on $\mathcal{H}$ in which paths P and Q are adjacent iff $|E(P) \setminus E(Q)| = |E(Q) \setminus E(P)| = 1$. Use H to prove that the number of paths in $\mathcal{H}$ that end in y is even. Deduce that in a graph without vertices of even degrees, every edge lies on an even number of Hamilton cycles.

Problem 6.5.

(a) Given a graph G, let H be a subgraph of G with the largest average degree. Prove that $\delta(H) \geq \frac{1}{2}d(G)$.

(b) For every positive integer k, find a function $f_k(n)$ that is linear in n such that $\mathrm{ex}(n, T) \leq f_k(n)$ for every n and every tree T with k edges.

Problem 6.6. Show that Hadwiger's Conjecture holds for line graphs; in other words, prove that if $\chi'(G) \geq r \geq 1$ for a graph G, then $L(G)$ contains a K^r-minor.

Hint. Assuming minimality might help.