Probabilistic method in combinatorics and algorithmics
WS 2016/17

Exercise sheet 1
Exercises for the exercise session on 17/10/2016

Problem 1.1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a discrete probability space and let A, B, C and $A_1, A_2, \ldots$ be elements of $\mathcal{A}$.

(a) Prove that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. What is the corresponding formula for $\mathbb{P}(A \cup B \cup C)$?

(b) Prove that $\mathbb{P}(\bigcup_i A_i) \leq \sum_i \mathbb{P}(A_i)$, provided that $\bigcup_i A_i \in \mathcal{A}$. Give an example of a probability space and sets $A_1, A_2, \ldots$ for which $\bigcup_i A_i \notin \mathcal{A}$.

Problem 1.2. Let X, Y be random variables and let $\alpha, \beta \in \mathbb{R}$. Prove that

$$\mathbb{E}[\alpha X + \beta Y] = \alpha \mathbb{E}[X] + \beta \mathbb{E}[Y] \qquad \text{and} \qquad |\mathbb{E}[X]| \leq \mathbb{E}[|X|].$$

Also prove that $\mathbb{E}[X] \geq \mathbb{E}[Y]$ if $X \geq Y$.

Problem 1.3. Given a discrete probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let $A \in \mathcal{A}$ with $\mathbb{P}(A) > 0$ and let X, Y be independent random variables.

(a) Define $\mathcal{A}'$ to be the collection of all subsets of A that lie in $\mathcal{A}$. For such a set B, set $\mathbb{P}'(B) := \mathbb{P}(B \mid A)$. Prove that $(A, \mathcal{A}', \mathbb{P}')$ is a discrete probability space.

(b) Prove that $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ and $\mathbb{E}[X \mid Y = y] = \mathbb{E}[X]$ for every $y \in Y(\Omega)$.

Problem 1.4. Let X, Y be random variables that only take non-negative values. Prove that

$$\mathbb{E}[(X+Y)^k] \geq \mathbb{E}[X^k] + \mathbb{E}[Y^k]$$

for every $k \in \mathbb{N}$ and give an example (with neither X nor Y being zero with probability one) where equality holds.

Problem 1.5. Let $n \in \mathbb{N}$ and let $\mathcal{F}$ be an inclusion-free family of subsets of $[n] := \{1, 2, \ldots, n\}$ (*inclusion-free* means that no element of $\mathcal{F}$ is a proper subset of another element). Choose a permutation σ of $[n]$ uniformly at random and define the random variable

$$X := |\{k \mid \{\sigma(1), \sigma(2), \ldots, \sigma(k)\} \in \mathcal{F}\}|.$$

Consider $\mathbb{E}[X]$ in order to prove that $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Problem 1.6. A *matching* in a graph is a set of edges such that no two share an end vertex; a *perfect matching* is a matching that covers all vertices. Determine the asymptotic probability (for $n \to \infty$) that $G(2n, \frac{1}{n})$ has a perfect matching.
Reminder. $G(n, p)$ is the random graph on the vertex set $[n]$ in which every edge is present with probability p, independently from each other.