Probabilistic method in combinatorics and algorithmics
WS 2016/17

Exercise sheet 2
Exercises for the exercise session on 27/10/2016

Problem 2.1. Let $k, r \in \mathbb{N}$ with $r \geq k \geq 2$ be given. An *r-uniform hypergraph* H consists of a set V of vertices and a set $E \subseteq \binom{V}{r}$ of edges (i.e. a 2-uniform hypergraph is just a graph).

(a) Prove that if $|E| \leq k^{r-1}$, then the vertices of H can be coloured with k colours so that no edge is monochromatic.

(b) Let

$$d_{r,k} := \sum_{i=1}^{k-1} \binom{k}{i} (-1)^{k-1-i} \frac{i^r}{k^r}$$

and prove that if $|E| \leq 1/d_{r,k}$, then the vertices of H can be coloured with k colours so that each edge contains all k colours.

Problem 2.2. Let $k \geq 3$. By $R(k, k)$ we denote the smallest number n such that every graph on n vertices contains a clique of size k or a stable set of size k. Prove that $R(k, k) = \omega\left(2^{\frac{k}{2}}\right)$ (that is, $R(k, k)/2^{\frac{k}{2}} \to \infty$ for $k \to \infty$) and that for every $\alpha \in \mathbb{R}_{>0}$ we have

$$R(k, k) > \left\lfloor \alpha 2^{\frac{k}{2}} - \alpha^k \right\rfloor .$$

Problem 2.3. Let x be a real number and let $n \geq 1$ be an integer.

(a) Show that $1 + x \leq e^x$ and that, if furthermore $x \geq 0$, then $1 + x \geq \exp(x - \frac{x^2}{2})$.

(b) Use part (a) to determine an upper and a lower bound for

$$\left(1 - \frac{1}{n+3}\right)^{2n+1} \cdot \left(1 + \frac{2}{3n-1}\right)^{3n-2} . \tag{1}$$

As a comparison, implement $1 + x = \exp(x - \frac{x^2}{2} + o(x^3))$ into (1) and compare the three results for $n \to \infty$.

Note. You will have to sum several rational functions in n. Feel free to use a computer for these calculations.

Problem 2.4. Let $n \geq k \geq 1$ be integers.

(a) Prove that

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} < \left(\frac{en}{k}\right)^k.$$

(b) Show that the falling factorial $(n)_k := \frac{n!}{(n-k)!}$ satisfies

$$(n)_k \leq n^k \exp\left(-\frac{k(k-1)}{2n}\right).$$

Also show that if $n > k$, then

$$\frac{\sqrt{2\pi}}{e} n^k \exp\left(-\frac{k^2}{2(n-k)}\right) \leq (n)_k\,.$$

For the last part, you might use the inequalities

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leq n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n.$$

Problem 2.5. Suppose that $A = (a_1, \dots, a_k) \subseteq [n]$ is such that for all subsets $I \neq J$ of A, the sum of the numbers in I is different from that of the numbers in J. Prove that for every $\alpha > 1$, we have

$$1 - \frac{1}{\alpha^2} \leq \frac{1}{2^k}\left(\alpha\sqrt{a_1^2 + \cdots + a_k^2} + 1\right).$$

Use this fact to deduce that there exist constants $c, C \in \mathbb{R}_{>0}$ such that

$$n \geq c\frac{2^k}{\sqrt{k}} \qquad \text{and} \qquad k \leq \log_2 n + \frac{1}{2}\log_2\log_2 n + C.$$

Problem 2.6. Let m, n be positive integers and consider the following random experiment. Given m balls and n bins, place each ball in a bin chosen uniformly at random, independently for each ball. For $\varepsilon > 0$, prove that

$$\mathbb{P}(\exists \text{ an empty bin}) \xrightarrow{n\to\infty} \begin{cases} 1 & \text{if } m \leq (1-\varepsilon)n\ln n, \\ 0 & \text{if } m \geq (1+\varepsilon)n\ln n. \end{cases}$$