Probabilistic method in combinatorics and algorithmics
WS 2016/17

Exercise sheet 3

Exercises for the exercise session on 7/11/2016

Problem 3.1. Let m, n be positive integers and consider the following random experiment. Given m balls and n bins, place each ball in a bin chosen uniformly at random, independently for each ball. For $\varepsilon > 0$, prove that

$$\mathbb{P}(\exists \text{ an empty bin}) \overset{n\to\infty}{\longrightarrow} \begin{cases} 1 & \text{if } m \leq (1-\varepsilon)n\ln n, \\ 0 & \text{if } m \geq (1+\varepsilon)n\ln n. \end{cases}$$

Problem 3.2. Let $n \geq r \geq 2$ and $p \in [0,1]$. Consider the random r-uniform hypergraph $H_r(n,p)$ on n vertices, in which each r-tuple forms an edge with probability p independently. Let A be the event that each $(r-1)$-tuple is contained in at least one edge. For $\varepsilon > 0$, prove that

$$\mathbb{P}(A) \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } p \leq (1-\varepsilon)\frac{(r-1)\ln n}{n}, \\ 1 & \text{if } p \geq (1+\varepsilon)\frac{(r-1)\ln n}{n}. \end{cases}$$

Problem 3.3. Let $A_1, \ldots, A_n$ be events in a probability space and let $D = ([n], E)$ be a dependency graph for the events $A_1, \ldots, A_n$. Use the Lovász Local Lemma to prove that the intersection of the events $\overline{A_1}, \ldots, \overline{A_n}$ has a positive probability if one of the following holds.

(a) For every $i = 1, \ldots, n$,

$$\mathbb{P}[A_i] < 1 \qquad \text{and} \qquad \sum_{(i,j)\in E} \mathbb{P}[A_j] \leq \frac{1}{4}.$$

Hint. Try $x_i = c\mathbb{P}[A_i]$ for a constant c.

(b) (Symmetric LLL) There exist $d \geq 1$ and $p \in [0,1]$ such that

- no vertex in D has more than d outgoing edges,
- $\mathbb{P}[A_i] \leq p$ for all $i = 1, \ldots, n$, and
- $ep(d+1) \leq 1$.

Problem 3.4. Let r, D be positive integers and $V_1, \ldots, V_r$ be disjoint sets of size $\lceil 2eD \rceil$. Suppose that G is a graph with vertex set $V = V_1 \cup \cdots \cup V_r$ such that every vertex of G has degree at most D. Use the Lovász Local Lemma to prove that there exists an *independent transversal*, that is, an independent set that contains precisely one vertex from each V_i.

Problem 3.5. Let β be a positive integer and let G be a graph with maximum degree $\Delta \geq \beta^\beta$. Prove that the vertices of G can be coloured with $Q = \left\lceil 16\Delta^{1+\frac{1}{\beta}} \right\rceil$ colours so that

- no two adjacent vertices have the same colour and
- for every vertex v, each colour appears at most β times in the neighbourhood of v.

Hint. Colour the vertices randomly. For every edge $e = uv$, the event A_e that u and v have the same colour is 'bad'. Similarly, for every set S of $\beta + 1$ vertices in the neighbourhood of a vertex v, the event A_S is 'bad'. Given a 'bad' event, with how many events of type A_e and A_S can there be dependencies, respectively? Try to apply the result from Problem 3.3a.