Probabilistic method in combinatorics and algorithmics
WS 2016/17

Exercise sheet 4

Exercises for the exercise session on 17/11/2016

Problem 4.1. Let r, d be positive integers and $V_1, \ldots, V_r$ be disjoint sets of size $\lceil 2ed \rceil$. Suppose that G is a graph with vertex set $V = V_1 \cup \cdots \cup V_r$ such that every vertex of G has degree at most d. Use the Lovász Local Lemma to prove that there exists an *independent transversal*, that is, an independent set that contains precisely one vertex from each V_i.

Problem 4.2. Let A_1, A_2, A_3 be events in a probability space such that the following holds.

- A_1 and A_2 are independent;
- A_2 and A_3 are independent;
- $A_1 \cap A_2$ and A_3 are independent.

Prove that A_1 and $A_2 \cap A_3$ are independent as well. Deduce from this that one may assume in the Lovász Local Lemma that the dependency graph D has the following property.

> Every vertex i of D is either isolated (i.e. there are no edges involving i) or has at least one outgoing edge (i.e. an edge (i, j)).

Problem 4.3. For the following probability spaces and sets of 'bad' events $A_1, \ldots, A_n$, prove that the Lovász Local Lemma *cannot* be applied (regardless of the choice of the dependency graph D).

(a) Let Ω be the set of all 01-sequences of length n with an even number of 1s and choose a sequence $s_1, \ldots, s_n$ from Ω uniformly at random. For each $i \in [n]$, denote by A_i the event that $s_i = 1$.

(b) Let Ω be the set of all permutations of $[n]$. We choose a permutation σ uniformly at random and denote, for each $i \in [n]$, by A_i the event that $\sigma(i) = i$.

Problem 4.4. Let $A_1, \ldots, A_n$ be events in a probability space and let $D = ([n], E)$ be a directed graph. We say that D is a *negative dependency graph* for $A_1, \ldots, A_n$ if the following holds

> For every $J \subseteq [n]$ and $i \in [n] \setminus J$, if $(i, j) \notin E$ for all $j \in J$ and $\mathbb{P}\left(\bigwedge_{j \in J} \overline{A_j}\right) > 0$, then $\mathbb{P}\left(A_i \,\middle|\, \bigwedge_{j \in J} \overline{A_j}\right) \leq \mathbb{P}(A_i)$.

Replacing 'dependency graph' by 'negative dependency graph' in the statement of the Lovász Local Lemma results in the so-called *Lopsided Lovász Local Lemma*.

(a) Sketch how the proof of the Lovász Local Lemma from the lecture generalises to a proof of the Lopsided Lovász Local Lemma.

(b) Follow the arguments sketched below to prove in that Problem 4.3b, the edgeless graph on $[n]$ is a negative dependency graph.

It suffices to show (why?) that

$$\mathbb{P}\left(\bigwedge_{j\in J}\overline{A_j}\;\middle|\;A_i\right)\leq\mathbb{P}\left(\bigwedge_{j\in J}\overline{A_j}\right)$$

for all $J\subset[n]$ and $i\in[n]\setminus J$ such that $\mathbb{P}(A_i)>0$. Show that this is equivalent to

$$n\left|A_i\wedge\bigwedge_{j\in J}\overline{A_j}\right|\leq\left|\bigwedge_{j\in J}\overline{A_j}\right|.$$

To prove this inequality, for each permutation $\sigma\in A_i\wedge\bigwedge_{j\in J}\overline{A_j}$, define n permutations $\sigma_1,\ldots,\sigma_n\in\bigwedge_{j\in J}\overline{A_j}$ and prove that $\sigma_k\neq\tau_l$ as soon as $k\neq l$ or $\sigma\neq\tau$.

Problem 4.5. Let $X_1,\ldots,X_n$ be independent random variables. Let $\lambda,t>0$ and $p\in[0,1]$ be given.

(a) Let $X_i\sim\mathrm{Be}(p)$ for all i, i.e. $\mathbb{P}(X_i=1)=p$ and $\mathbb{P}(X_i=0)=1-p$. We consider the random variables $X=\sum_i X_i\sim\mathrm{Bin}(n,p)$ and $Y\sim\mathrm{Poi}(\lambda)$, where the latter is defined by $\mathbb{P}(Y=k)=\frac{\lambda^k}{k!}e^{-\lambda}$.

Determine the variances of X and Y.

(b) Suppose that each X_i only takes values in $[0,1]$ and write $\sigma^2=\mathrm{Var}[X]$. Prove that

$$\mathbb{P}\big(X\geq\mathbb{E}[X]+t\big)<\exp\left(-\frac{t^2}{2\left(\sigma^2+\frac{t}{3}\right)}\right).$$

Hint. In the lecture it was mentioned that

$$\varphi(y):=(1+y)\ln(1+y)-y\geq\frac{y^2}{2\left(1+\frac{y}{3}\right)}$$

for $y\geq 0$. Determine values for α and y so that $\exp(-\alpha\varphi(y))$ is exactly the desired upper bound and try to prove that the probability is smaller than $\exp(-\alpha\varphi(y))$.

To this end, start as in the proof of the first Chernoff bound from the lecture. Bound each factor $\mathbb{E}[e^{uX_i}]$ by a term $\exp((e^u-1)\mathbb{E}[X_i])$ (why is this an upper bound?) and then implement a suitable value for u.