Probabilistic method in combinatorics and algorithmics
WS 2016/17

Exercise sheet 5

Exercises for the exercise session on 12/12/2016

Problem 5.1. Let $k \geq 2$ be an integer and let $\delta > 0$ be given. For each $i = 1, \ldots, k$, we have a coin C_i that shows 'head' with probability $p_i \in (0,1)$. However, we cannot distinguish the coins.

(a) Suppose that $k = 2$ and we know the values p_1, p_2, where $p_1 < p_2$. Consider the following algorithm to identify C_1.

Algorithm A: Pick a random coin and toss it n times. If 'head' comes up less than $\frac{p_1+p_2}{2}n$ times, then pick this coin, otherwise pick the other coin.

Prove that Algorithm A succeeds in identifying C_1 with probability larger than $1-\delta$, provided that $n > \frac{8p_2}{(p_2-p_1)^2} \ln\left(\frac{1}{\delta}\right)$.

(b) Write $p := \min\{p_1, \ldots, p_k\}$ and consider the following algorithm.

Algorithm B: Toss all coins n times each and pick a coin that showed 'head' the least number of times.

Prove that with probability larger than $1-\delta$, Algorithm B picks a coin C_j with $p_j \leq 2p$, provided that $n > \frac{16}{p} \ln\left(\frac{k}{\delta}\right)$.

Problem 5.2. Let $\varepsilon > 0$ and consider the graph $G = G(n,p)$ with $p = \frac{1}{\sqrt{n}}$. Denote by T the number of triangles in G and by $d(v)$ the degree of a vertex v. Use the Chernoff bounds to prove that with probability $1 - o(1)$ we have

$$\left|d(v) - \sqrt{n}\right| < \left(\sqrt{2} + \varepsilon\right) (\ln(n))^{1/2} n^{1/4} \tag{1}$$

for every vertex v as well as

$$T \in \left[\frac{1-\varepsilon}{6} n^{3/2}, \frac{1+\varepsilon}{6} n^{3/2}\right]. \tag{2}$$

Hint. For (2), it might help to first count all triangles that include a fixed vertex.

Problem 5.3. Consider the following more general version of Azuma's inequality. Let $X_0, \ldots, X_m$ be a martingale for which there exist positive constants $c_1, \ldots, c_m$ such that

$$|X_i - X_{i-1}| \leq c_i \quad \forall 1 \leq i \leq m.$$

Then for every $t > 0$

$$\mathbb{P}(X_m - X_0 > t) < \exp\left(-\frac{t^2}{2\sum_{i=1}^m c_i^2}\right), \tag{3}$$

$$\mathbb{P}(X_m - X_0 < -t) < \exp\left(-\frac{t^2}{2\sum_{i=1}^m c_i^2}\right). \tag{4}$$

(a) Show that this implies the 'basic' version of Azuma's inequality that was proved in the lecture. Vice versa, suppose that $X_0, \ldots, X_m$ satisfies the conditions above and define from this a martingale $\tilde{X}_0, \ldots, \tilde{X}_m$ to which the basic version of Azuma's inequality can be applied. Argue why we cannot directly use the result of this application of Azuma's inequality to derive (3).

(b) Prove the general version of Azuma's inequality by following the proof from the lecture.

Hint. Only α and the function h have to be chosen differently.

Problem 5.4. Prove that the edge exposure martingale and the vertex exposure martingale are indeed martingales (regardless of the function f). Also prove that if f satisfies the edge (resp. vertex) Lipschitz condition, then the corresponding edge (resp. vertex) exposure martingale satisfies $|X_{i+1} - X_i| \leq 1$ for all i.

Problem 5.5. For a positive integer n, let G be the graph with vertex set $(\mathbb{Z}_7)^n$ and an edge between two vertices u and v if and only if they differ in precisely one coordinate. Suppose we are given a constant $c > 0$ and a set U of 7^{n-1} vertices. Denote by W the set of vertices of G that have distance more than $(2+c)\sqrt{n}$ from U. (The distance from a set of vertices is the minimum of the distances from the vertices in that set.) Show that

$$|W| \leq 7^n e^{-\frac{c^2}{2}}.$$