Probabilistic method in combinatorics and algorithmics
WS 2016/17

Exercise sheet 6

Exercises for the exercise session on 16/01/2017

Problem 6.1. At a Christmas party, there are n people. For each person, there is precisely one present with their name written on it. The presents are now distributed as follows. First, everyone gets a random present. If a person happens to receive the present with their name on it, they keep the present. All other presents are returned and then distributed randomly again. This continues until everyone has the right present.
Denote by Y_i the number of people who do not have the right present yet after i rounds of distributing. Prove that $Y_0, Y_1, \dots$ is *not* a martingale, but $X_0, X_1, \dots$ with

$$X_0 = Y_0, \qquad X_i = \begin{cases} Y_i + i & \text{if } Y_{i-1} > 0, \\ X_{i-1} & \text{otherwise} \end{cases} \quad \text{for } i \geq 1$$

is. Find a real number $\alpha > 0$ so that the process ends after at most αn rounds with probability at least $\frac{1}{2}$.
Hint. We cannot apply Azuma's inequality directly, because $|X_{i+1} - X_i| \leq 1$ does not hold. But maybe $X_{i+1} - X_i \leq 1$ is enough to have at least one of the bounds?

Problem 6.2. Let n be a positive integer and let $p = p(n) \in (0, 1)$. Let X be the sum of n i.i.d. random variables $Y_1, \dots, Y_n$, which are 1 with probability p and 0 with probability $1-p$. Define a martingale $X_0, \dots, X_n$ that satisfies $X_0 = \mathbb{E}[X]$ and $X_n = X$. Compare the bound that Azuma's inequality gives for

$$\mathbb{P}(X > \mathbb{E}[X] + t)$$

with the bound from Chernoff's inequality. Which one is better? Does the answer depend on the choice of $p(n)$ and t?

Problem 6.3. Let $k \geq 3$ be a constant. For $n \geq k$ (where we think of n as tending to ∞), set

$$\mu := \binom{n}{k} 2^{-\binom{k}{2}}.$$

Prove that there exists a function $f(n) = o(1)$ such that

$$\binom{n}{k} \sum_{i=2}^{k-1} \binom{k}{i} \binom{n-k}{k-i} 2^{-\binom{k}{2}} 2^{-\left(\binom{k}{2} - \binom{i}{2}\right)} \leq (1 + f(n)) \frac{k^4}{n^2} \mu^2$$

as $n \to \infty$.
Hint. Remember that k is constant, but we can choose n as large as necessary. If n becomes large, is there one summand that is much larger than all the others (even than all the others together)?

Problem 6.4. A famous result about random graphs states that $G(n,p)$ has a perfect matching with probability $1-o(1)$ whenever n is even and $p \geq \frac{(1+\varepsilon)\ln(n)}{n}$. (Note that this is just large enough in order to guarantee that there are no isolated vertices.) Prove the following weaker result.
Denote by $m(G)$ the size of the largest matching in the graph G and write $\mu := \mathbb{E}[m(G(n,p))]$. Suppose that $p = p(n) \in [0,1]$ is such that $pn \to \infty$. Prove that for every $t > 0$,

$$\mathbb{P}(m(G(n,p)) \leq \mu - t\sqrt{n-1}) < \exp\left(-\frac{t^2}{2}\right)$$

and show that for every $\varepsilon > 0$,

$$\mu \geq (1-\varepsilon)\frac{pn}{2}$$

if n is large enough.

Problem 6.5. Let $\varepsilon > 0$. Use Janson's inequality to prove that there exists n_0 so that for every integer $n \geq n_0$, there is a graph on n vertices that contains *every* graph on at most $(2-\varepsilon)\log_2 n$ vertices as an induced subgraph.
Hint. Any ingredients of the applications of Janson's inequality from the lecture can be used without reproving them.