Advanced and algorithmic graph theory

Summer term 2018

Exercise sheet 1

Exercises for the exercise session on 15/03/2018

Problem 1.1. Let T be a spanning tree of a connected graph G.
A *fundamental circuit* of G (with respect to T) is the edge set of a cycle in G that contains precisely one edge that is not in T.
For each edge $e \in E(T)$, the graph $T - e$ has precisely two components, with vertex sets A_e and B_e, say. The set of all edges of G between A_e and B_e is called a *fundamental cut* of G (with respect to T).
Prove that the following statements are equivalent for a set $F \subseteq E(G)$.

(i) F is the symmetric difference of fundamental circuits (w.r.t. T);

(ii) Every vertex in the graph $(V(G), F)$ has even degree;

(iii) F meets each fundamental cut (w.r.t. T) in an even number of edges.

Problem 1.2. Let G be a connected graph. Prove that the following statements are equivalent for an edge $e \in E(G)$.

(i) $G - e$ is not connected;

(ii) No cycle in G contains e;

(iii) Every spanning tree of G contains e;

(iv) Every spanning tree of G found by an application of BFS(G,u) contains e (independently of the choice of u and of the order in which we check the neighbours of a vertex in the FOR-loop);

(v) Every spanning tree of G found by an application of DFS(G,u) contains e (independently of the choice of u and of the order in which we check the neighbours of a vertex in the FOR-loop).

Problem 1.3. Let G be a connected weighted graph. For $i = 1, 2$, denote by T_i the graph that is generated by Algorithm i from the lecture. Reminder:

Alg. 1: Recursively add edges of smallest weight that do not create cycles.

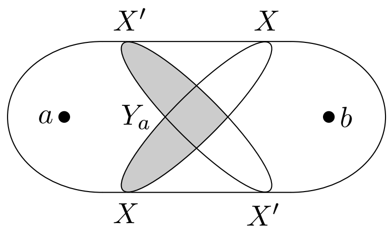
Alg. 2: Recursively delete edges of largest weight that do not separate.

Prove that both T_1 and T_2 are spanning trees of G and that both have smallest total weight among all spanning trees of G.

Problem 1.4. Let G be a graph and let a, b be two distinct vertices of G. Suppose that each of the vertex sets $X, X' \subseteq V(G) \setminus \{a, b\}$ is an a-b separator. Denote by C_a and C_b the component of $G - X$ that contains a and b, respectively. Define C'_a and C'_b analogously for X'. Prove that the sets

$$
\begin{aligned}
Y_a &:= (X \cap C'_a) \cup (X \cap X') \cup (X' \cap C_a), \\
Y_b &:= (X \cap C'_b) \cup (X \cap X') \cup (X' \cap C_b)
\end{aligned}
$$

are a-b separators as well.



Problem 1.5. Let G, a, b, X, X', Y_a, Y_b be as in Problem 1.4.

(a) Prove that X is a minimal a-b separator (with respect to containment, i.e. if we remove a vertex from X, then the remaining set is no longer an a–b separator) if and only if each vertex in X has neighbours in both C_a and C_b.

(b) Suppose that both X and X' have smallest size among all a–b separators in $V \setminus \{a, b\}$. Prove that Y_a and Y_b have the same size as X and X'.

(c) Give an example for which X and X' are minimal a-b separators (w.r.t. containment), but Y_a and Y_b are not minimal.

Problem 1.6. Let G be k-connected, where $k \geq 2$. Prove that for each set of k vertices in G, there is a cycle in G that contains them all.