Advanced and algorithmic graph theory

Summer term 2018

Exercise sheet 3

Exercises for the exercise session on 26/04/2018

Problem 3.1. Let G be bipartite with sides A and B and suppose that each vertex v has a preference order $\leq_v$ on its set of neighbours. A stable matching M is called *A-optimal* if each vertex $a \in A$ is matched to its "best" neighbour among all vertices that are possible partners for a in stable matchings. (Formally: If $ab \in M$ and $ab' \in M'$ for some stable matching M', then $b \leq_a b'$.) We define *B-optimal*, *A-pessimal* (worst possible for A) and *B-pessimal* analogously.
Prove that a stable matching is A-optimal if and only if it is B-pessimal. Furthermore, prove that every bipartite G has a unique A-optimal stable matching.

Problem 3.2. Show that the stable matching generated by STABLE is A-optimal (and thus also B-pessimal).
Hint. Consider, for contradiction, the first time in STABLE that some vertex in A is rejected by its matching partner in the A-optimal stable matching.

Problem 3.3. Prove that Tutte's Theorem (Theorem 2.8 from the lecture) implies Hall's Theorem (Theorem 2.2) and that Theorem 2.9 by Gallai and Edmonds implies König's Theorem (Theorem 2.1).
Hint. For the first part, start by showing that the case $|A| = |B|$ of Hall's theorem implies the general case. In order to show that the Tutte condition holds in this special case (provided that the marriage condition holds), distinguish the odd components of $G - S$ depending on whether they contain more vertices from A or from B.

Problem 3.4. A graph G is called *transitive* if for every two vertices $u, v \in V(G)$, there is an automorphism $\varphi\colon V(G) \to V(G)$ (that is, φ is bijective and $xy \in E(G)$ if and only if $\varphi(x)\varphi(y) \in E(G)$) that maps u to v.
Show that every connected transitive graph with an even number of vertices has a perfect matching.

Problem 3.5. Prove Proposition 3.4. from the lecture: The following statements are equivalent for every plane graph G on $n \geq 3$ vertices.

(i) G is maximally planar;

(ii) G is maximally plane;

(iii) G is a triangulation;

(iv) $\|G\| = 3n - 6$.

Problem 3.6. Prove Corollary 3.8. from the lecture: Every maximally planar graph G with at least four vertices is 3-connected.
Hint. Suppose, for contradiction, that $\{u, v\}$ is a separator of size two. If $uv \in E(G)$, derive a contradiction to the maximality of G. Otherwise, use the non-planarity of $G + uv$ to obtain a contradiction to the planarity of G.