Advanced and algorithmic graph theory

Summer term 2018

Exercise sheet 4

Exercises for the exercise session on 17/05/2018

Problem 4.1. A graph is called *outerplanar* if it is planar and has a drawing in which all vertices lie on the boundary of the outer face. Prove that the following statements are equivalent for a graph G.

(i) G is outerplanar;

(ii) G contains neither K^4 nor $K_{2,3}$ as a minor;

(iii) G contains neither K^4 nor $K_{2,3}$ as a topological minor.

Problem 4.2. Prove that for every closed surface, the set of forbidden topological minors is finite.
Hint. Start with the set of forbidden minors. For each forbidden minor H, find a finite number of graphs so that every graph with an MH contains a subdivision of (at least) one of them. Taking a look at how we found a TK^5 or a $TK_{3,3}$ in an MK^5 or $MK_{3,3}$ in the lecture might help.

Bonus problem. Find an infinite set $\{G_1, G_2, \dots\}$ of finite graphs in which no G_i is a topological minor of G_j for $i \neq j$.
Hint. Try to construct G_{i+1} from G_i so that the vertices of "large" degrees in G_{i+1} are arranged in a way that makes it impossible to find a TG_i in G_{i+1}.

Problem 4.3. Describe a planarity recognition algorithm along the following lines.

- Reduce the problem to 2-connected graphs G with a linear number of edges. (How?)
- Find a cycle C in G for which every attachment set of a fragment is independent in C. (How?)
- Recursively check planarity of $F \cup C$ for each fragment F.
- Check whether $O_G(C)$ is bipartite. (How?)

What running time can you achieve? Assuming that we are given a cycle C with $V(C) = V(G)$, how much does the running time improve?
Note. Do not spend too much time trying to optimise the running time. Rough estimates are enough.

Problem 4.4. Find all mistakes in the following "proof" of the Four Colour Theorem. (In other words, point out which arguments are valid and which are false.)

Suppose, for contradiction, that the Four colour theorem is false. Let v be a vertex of degree $d := \delta(G) \leq 5$ in a smallest non-4-colourable planar graph G. Fix a drawing of G and a 4-colouring c of $G - v$. Denote the neighbours of v by $x_1, \ldots, x_d$ in the order they lie around v in the drawing. Furthermore, set $G_{i,j} := G[c^{-1}(i) \cup c^{-1}(j)]$. Since G is not 4-colourable, we know that

$$\text{no 4-colouring of } G - v \text{ uses less than four colours for } N(v). \tag{1}$$

In particular, $d \geq 4$. W.l.o.g. $c(x_i) = i$ for $i = 1, 2, 3, 4$ and, if $d = 5$, $c(x_5) \in \{1, 2\}$.

Suppose first that $d = 4$. If there is no x_1–x_3 path in $G_{1,3}$, then we can recolour x_1 with colour 3 by exchanging the colours in the component of $G_{1,3}$ that contains x_1 and obtain a colouring c' that contradicts (1). Otherwise, we can recolour x_2 with colour 4 analogously.

Now suppose that $d = 5$ and $c(x_5) = 1$. If there is no x_3–$\{x_1, x_5\}$ path in $G_{1,3}$, we can recolour x_3 with colour 1. Otherwise, we can recolour x_2 with colour 4, again yielding a contradiction.

Finally, suppose that $d = 5$ and $c(x_5) = 2$. If there is no x_1–x_3 path in $G_{1,3}$ or no x_1–x_4 path in $G_{1,4}$, then we can recolour x_1 with colour 3 or 4, respectively. Otherwise, there is neither an x_2–x_4 path in $G_{2,4}$ nor an x_5–x_3 path in $G_{2,3}$. Thus we can recolour x_2 with colour 4 and x_5 with colour 3, again a contradiction to (1).

Problem 4.5. Let G be a graph.

(a) Show that there exists an ordering σ_0 of $V(G)$ such that $\chi_{\text{Gr}}(G, \sigma_0) = \chi(G)$.

(b) Prove that $\chi_{\text{Gr}}(G, \sigma) \leq \frac{1}{2} + \sqrt{2\,\|G\| + \frac{1}{4}}$ for *every* ordering σ of $V(G)$.

(c) Construct, for every positive integer n, a graph G_n on $2n$ vertices and an ordering σ_1 of $V(G_n)$, for which $\chi(G_n) = 2$, but $\chi_{\text{Gr}}(G_n, \sigma_1) = n$.

Problem 4.6. Prove that the upper bound $1 + \max_{H \subseteq G} \delta(H)$ for $\chi(G)$ is strictly larger than $1 + \frac{1}{2}d(G)$.